

TAMENESS OF RIEMANNIAN LOCALLY SYMMETRIC SPACES ARISING FROM ANOSOV REPRESENTATIONS

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ABSTRACT. We construct compactifications of Riemannian locally symmetric spaces arising as quotients by Anosov representations. These compactifications are modeled on generalized Satake compactifications and, in certain cases, on maximal Satake compactifications. We deduce that these Riemannian locally symmetric spaces are topologically tame, i.e. homeomorphic to the interior of a compact manifold with boundary. We also construct domains of discontinuity (not necessarily with a compact quotient) in a much more general setting.

1. INTRODUCTION

Any discrete subgroup Λ of a semisimple (or reductive) Lie group G acts properly discontinuously by isometries on the Riemannian symmetric space $X = G/K$ of G . The quotient space $M = \Lambda \backslash X$ is a Riemannian locally symmetric orbifold, which is noncompact except if Λ is a uniform lattice in G . When M has finite volume (i.e. Λ is a lattice), compactifications of M have been well studied: see [BJ06] for an overview of various compactifications with their properties and uses. When M has infinite volume, compactifications of M have been mainly studied in the case that G has real rank one, i.e. that X is a negatively curved manifold. In this case, compactifications of M have been constructed for geometrically finite representations (see [Ji05, Prop. 3.5], based on [AX04, Th. 6.5]). Recently there has been a growing interest in Zariski-dense subgroups of semisimple Lie groups, also of higher rank, which are not lattices, i.e. for which M has infinite volume. However, when G has higher real rank and Λ has infinite covolume, compactifications of M are not well studied, and very little is known.

In this paper we construct compactifications of M when Λ is the image of an Anosov representation. When G has real rank one, the images of Anosov representations are exactly the convex cocompact subgroups; when G has higher real rank, images of Anosov representations provide a meaningful

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generalization of convex cocompact subgroups [Lab06, GW12, KLPa, KLPb, KLPc].

Theorem 1.1. *Let $X = G/K$ be a Riemannian symmetric space, where G is a noncompact real semisimple Lie group and K a maximal compact subgroup of G . Let Γ be a word hyperbolic group and P a proper parabolic subgroup of G . For any P -Anosov representation $\rho \in \text{Hom}(\Gamma, G)$, the Riemannian locally symmetric space $\rho(\Gamma) \backslash X$ admits a compactification which is an orbifold with corners, locally modeled on a generalized Satake compactification of X .*

We introduce generalized Satake compactifications in Appendix A (Definition A.6). They provide a natural extension of the class of Satake compactifications, which satisfies the functorial property that the closure of a totally geodesic subsymmetric space $Y \subset X$ in a generalized Satake compactification of X is a generalized Satake compactification of Y . This is not true for Satake compactifications. In Theorem 1.1 the generalized Satake compactification *dominates* (i.e. admits a continuous G -equivariant map to) the maximal Satake compactification of X .

For specific Anosov representations, we can improve Theorem 1.1 and construct a compactification modeled on the maximal Satake compactification of X .

Theorem 1.2. *Let $X = G/K$ be a Riemannian symmetric space where G is a noncompact real semisimple Lie group. Then there exists a maximal proper parabolic subgroup P of G such that for any word hyperbolic group Γ and any P -Anosov representation $\rho : \Gamma \rightarrow G$, the Riemannian locally symmetric space $\rho(\Gamma) \backslash X$ admits a compactification which is an orbifold with corners, locally modeled on the maximal Satake compactification of X .*

For more precise statements, we refer to Theorem 5.6 in the case that G is simple and Theorem 5.8 in the general case.

- Remarks 1.3.*
- (1) If P' is a parabolic subgroup of G contained in P , then any P' -Anosov representation $\rho : \Gamma \rightarrow G$ is P -Anosov. In particular, Theorem 1.2 applies to any P' -Anosov representation with $P' \subset P$ (for instance to any P_{\min} -Anosov representation where P_{\min} is a minimal parabolic subgroup of G).
 - (2) For Anosov representations $\rho : \Gamma \rightarrow \text{O}(b)$ (resp. $\text{O}(b_{\mathbb{C}})$) into the orthogonal of a nondegenerate real (resp. complex) symmetric bilinear form b (resp. $b_{\mathbb{C}}$), we actually construct compactifications of the Riemannian locally symmetric spaces that are modeled on a minimal Satake compactification: see Theorems 4.1 and 5.9 for precise statements.
 - (3) In the preprint [KL], Kapovich and Leeb construct, by a different method, compactifications modeled on the maximal Satake compactification for Riemannian locally symmetric spaces arising from any Anosov representation. They also prove a converse statement: if a subgroup Λ of G is uniformly τ_{mod} -regular (a uniform version of the notion of P_{θ} -divergence to be found below in Section 3.1) and if the locally symmetric space $\Lambda \backslash X$ admits a compactification modeled on the maximal Satake compactification of X , then the group Λ is word hyperbolic and the inclusion of Λ in G is P_{θ} -Anosov.

The compactifications of $M = \Lambda \backslash X$ that we construct in Theorems 1.1 and 1.2 and their refinements (Theorems 4.1, 5.6, 5.8, and 5.9) are all obtained by considering a Satake or generalized Satake compactification \bar{X} of X , and removing from it a *bad set* \mathcal{N} , determined by the dynamical properties of sequences of elements of Λ , such that the action of Λ on $\bar{X} \setminus \mathcal{N}$ is properly discontinuous. The idea of describing \mathcal{N} in terms of dynamics of sequences is inspired by [Fra05].

In Theorem 5.4 we define a bad set \mathcal{N} in a compactification \bar{X} and obtain a properly discontinuous action on $\bar{X} \setminus \mathcal{N}$ for *any* discrete subgroup Λ of G . This yields a manifold with corners containing $\Lambda \backslash X$ as a dense subset.

For Anosov representations, the compactification \bar{X} and the bad set \mathcal{N} can be chosen in such a way that the quotient $\Lambda \backslash (\bar{X} \setminus \mathcal{N})$ is compact, providing a genuine compactification of $M = \Lambda \backslash X$. Let us emphasize that the topology on $\bar{X} \setminus \mathcal{N}$ is induced by the inclusion into \bar{X} . This is in contrast to the situation of Satake compactifications of Riemannian locally symmetric spaces of finite volume, where one takes the union of X with a subset of $\bar{X} \setminus X$, but changes the topology on the union. A combination of these two strategies might provide an approach to compactify Riemannian locally symmetric spaces of infinite volume that do not arise from Anosov representations, but from more general discrete subgroups.

We apply our construction of compactifications to prove topological tameness.

Theorem 1.4. *Let $X = G/K$ be a Riemannian symmetric space, where G is a noncompact real semisimple Lie group and K a maximal compact subgroup of G . Let Γ be a torsion-free word hyperbolic group and P a proper parabolic subgroup of G . For any P -Anosov representation $\rho \in \text{Hom}(\Gamma, G)$, the Riemannian locally symmetric space $\rho(\Gamma) \backslash G/K$ is topologically tame, i.e. homeomorphic to the interior of a compact manifold with boundary.*

Organization of the paper. In Section 2 we introduce some notation and recall some basic facts on semisimple Lie groups and their parabolic subgroups. In Section 3 we recall the notions of limit set and Anosov representation, and establish some useful properties. In Section 4 we prove Theorem 4.1, which gives compactifications modeled on minimal Satake compactifications for orthogonal groups. From this, in Section 5, we deduce Theorems 1.1 and 1.2 in full generality as well as Theorem 1.4; compactifications for complex orthogonal groups (Theorem 5.9) are also discussed. In Appendix A we give a description of Satake compactifications and a few properties of generalized Satake compactifications.

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2. BACKGROUND ON LIE GROUPS AND THEIR PARABOLIC SUBGROUPS

In this section we recall some basic facts on the structure of real reductive Lie groups and their parabolic subgroups.

Let G be a real reductive Lie group with Lie algebra \mathfrak{g} . In the whole paper, we assume G to be noncompact, equal to a finite union of connected components (for the real topology) of $\mathbf{G}(\mathbf{R})$ for some algebraic group \mathbf{G} .

2.1. Restricted roots. Let K be a maximal compact subgroup of G , with Lie algebra \mathfrak{k} , and let \mathfrak{a} be a maximal abelian subspace of the orthogonal complement of \mathfrak{k} in \mathfrak{g} for the Killing form κ . The *real rank* of G is by definition the dimension of \mathfrak{a} . Let Σ be the set of restricted roots of \mathfrak{a} in \mathfrak{g} , i.e. the set of nonzero linear forms $\alpha \in \mathfrak{a}^*$ for which

$$\mathfrak{g}_\alpha := \{z \in \mathfrak{g} \mid \text{ad}(a)(z) = \langle \alpha, a \rangle z \quad \forall a \in \mathfrak{a}\}$$

is nonzero. (We denote by $\langle \cdot, \cdot \rangle : \mathfrak{a}^* \times \mathfrak{a} \rightarrow \mathbf{R}$ the natural pairing.) Let $\Delta \subset \Sigma$ be a system of *simple restricted roots*, i.e. any element of Σ is expressed uniquely as a linear combination of elements of Δ with coefficients all of the same sign. Let

$$\bar{\mathfrak{a}}^+ := \{Y \in \mathfrak{a} \mid \langle \alpha, Y \rangle \geq 0 \quad \forall \alpha \in \Delta\}$$

be the closed positive Weyl chamber of \mathfrak{a} associated with Δ . The *restricted Weyl group* of \mathfrak{a} in \mathfrak{g} is the group $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$, where $N_K(\mathfrak{a})$ (resp. $Z_K(\mathfrak{a})$) is the normalizer (resp. centralizer) of \mathfrak{a} in K . There is a unique element $w_0 \in W$ such that $w_0 \cdot (-\bar{\mathfrak{a}}^+) = \bar{\mathfrak{a}}^+$; the involution of \mathfrak{a} defined by $Y \mapsto -w_0 \cdot Y$ is called the *opposition involution*. The corresponding dual linear map preserves Δ ; we shall denote it by

$$(2.1) \quad \begin{aligned} \mathfrak{a}^* &\longrightarrow \mathfrak{a}^* \\ \alpha &\longmapsto \alpha^* = -w_0 \cdot \alpha. \end{aligned}$$

2.2. Cartan decomposition. Recall that G admits the *Cartan decomposition* $G = K \exp(\bar{\mathfrak{a}}^+)K$: any $g \in G$ may be written

$$(2.2) \quad g = k_g a_g \ell_g$$

for some $k_g, \ell_g \in K$ and a unique a_g in $\exp(\bar{\mathfrak{a}}^+)$, $\mu(g) = \log a_g$ is called the Cartan projection of g (see [Hel01, Ch. IX, Th. 1.1]). This defines a proper, continuous, surjective map

$$\mu : G \longrightarrow \bar{\mathfrak{a}}^+$$

called the *Cartan projection*, inducing a homeomorphism $K \backslash G / K \simeq \bar{\mathfrak{a}}^+$. The pair (k_g, ℓ_g) is not unique, but is determined uniquely up to the action of the centralizer of $\mu(g)$ in K .

2.3. Parabolic subgroups, flag varieties and transversality. Let $\Sigma^+ \subset \Sigma$ be the set of *positive restricted roots* with respect to Δ , i.e. restricted roots that are nonnegative linear combinations of elements of Δ . For any nonempty subset $\theta \subset \Delta$, we denote by P_θ the normalizer in G of the Lie algebra $\mathfrak{u}_\theta = \bigoplus_{\alpha \in \Sigma^+ \setminus \text{span}(\Delta \setminus \theta)} \mathfrak{g}_\alpha$. Explicitly,

$$\text{Lie}(P_\theta) = \mathfrak{p}_\theta = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Sigma^+ \cap \text{span}(\Delta \setminus \theta)} \mathfrak{g}_{-\alpha}.$$

In particular, $P_\theta = G$ and P_Δ is a minimal parabolic subgroup of G .¹ Any parabolic subgroup of G is conjugate to P_θ for some $\theta \subset \Delta$.

The standard opposite parabolic subgroup to P_θ is the normalizer P_θ^- of $\mathfrak{u}_\theta^- = \bigoplus_{\alpha \in \Sigma^+ \setminus \text{span}(\Delta \setminus \theta)} \mathfrak{g}_{-\alpha}$. Note that P_θ^- is conjugate to P_{θ^*} . We shall consider the flag varieties

$$\begin{aligned}\mathcal{F}_\theta &= \{P \subset G \mid P \text{ is conjugate to } P_\theta\} \simeq G/P_\theta, \\ \mathcal{F}_{\theta^*} &= \{P \subset G \mid P \text{ is conjugate to } P_\theta^-\} \simeq G/P_\theta^- \simeq G/P_{\theta^*}.\end{aligned}$$

Definition 2.1. A pair $(P, Q) \in \mathcal{F}_\theta \times \mathcal{F}_{\theta^*}$ of parabolic subgroups of G is called *transverse* if $P \cap Q$ is a reductive Lie group, or equivalently if (P, Q) is conjugate to (P_θ, P_θ^-) under the diagonal action of G .

2.4. Example: general linear groups. Let $G = \text{GL}_{\mathbf{R}}(V)$, where V is a real vector space of dimension n . We may fix a basis (e_1, \dots, e_n) of V and take K to be $\text{O}(n)$ and \mathfrak{a} to be the space of diagonal matrices in that basis:

$$\mathfrak{a} = \{\text{diag}(\lambda_1, \dots, \lambda_n) \mid \lambda_1, \dots, \lambda_n \in \mathbf{R}\}.$$

Let $(\varepsilon_1, \dots, \varepsilon_n)$ be the standard basis of \mathfrak{a}^* , i.e. $\langle \varepsilon_i, \text{diag}(\lambda_1, \dots, \lambda_n) \rangle = \lambda_i$ for all i . The root system is

$$\Sigma = \{\varepsilon_i - \varepsilon_j \mid i \neq j, 1 \leq i, j \leq n\}.$$

A system of simple roots is

$$\Delta = \{\alpha_i \mid 1 \leq i \leq n-1\},$$

where $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$. The opposition involution switches α_i and α_{n-i} .

The parabolic subgroup $P_{\{\alpha_i\}}$ will be denoted P_i ; it is the stabilizer in $\text{GL}_{\mathbf{R}}(V)$ of the subspace $\mathbf{R}e_1 \oplus \dots \oplus \mathbf{R}e_i$ of V . The flag variety $\mathcal{F}_{\{\alpha_i\}} = \text{GL}_{\mathbf{R}}(V)/P_i$ identifies with the Grassmannian $\text{Gr}_i(V) \simeq \text{Gr}_{n-i}(V^*)$. In particular, $\mathcal{F}_{\{\alpha_1\}}$ identifies with the projective space $\mathbf{P}(V)$ and $\mathcal{F}_{\{\alpha_{n-1}\}}$ with the projective dual space $\mathbf{P}(V^*)$. The notion of transversality on $\text{Gr}_i(V) \times \text{Gr}_{n-i}(V)$ from Definition 2.1 is the natural one: a pair (W_i, W_{n-i}) is transverse if and only if $W_i \oplus W_{n-i} = V$.

2.5. Example: indefinite orthogonal groups. Let b be a nondegenerate bilinear symmetric form of signature (p, q) on a real vector space V . Suppose that $(p, q) \neq (1, 1)$ and that $p \geq q > 0$ (the case $q \geq p > 0$ is similar). Let G be the orthogonal group $\text{O}(b)$. There is a basis (e_1, \dots, e_{p+q}) of V such that for any $x = \sum_{i=1}^{p+q} x_i e_i$ and $y = \sum_{i=1}^{p+q} y_i e_i$,

$$b(x, y) = \sum_{i=1}^q (x_i y_{p+q-i+1} + x_{p+q-i+1} y_i) + \sum_{i=q+1}^p x_i y_i.$$

We may take $K = \text{O}(p+q) \cap G$, which is isomorphic to $\text{O}(p) \times \text{O}(q)$, and

$$\mathfrak{a} = \{\text{diag}(\lambda_1, \dots, \lambda_q, 0, \dots, 0, -\lambda_q, \dots, -\lambda_1) \mid \lambda_1, \dots, \lambda_q \in \mathbf{R}\}.$$

Let $(\varepsilon_1, \dots, \varepsilon_q)$ be the standard basis of \mathfrak{a}^* , i.e.

$$\langle \varepsilon_i, \text{diag}(\lambda_1, \dots, \lambda_q, 0, \dots, 0, -\lambda_q, \dots, -\lambda_1) \rangle = \lambda_i$$

¹This is the same convention as in [GGKWa, GGKWb], but the opposite convention to [GW12].

for all i . The restricted root system is

$$\begin{aligned}\Sigma &= \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq q\} \cup \{\pm\varepsilon_i \mid 1 \leq i \leq q\} \text{ if } p > q \quad (\text{type } B_q), \\ \Sigma &= \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq p\} \text{ if } p = q \quad (\text{type } D_p).\end{aligned}$$

A system of simple restricted roots is $\Delta = \{\alpha_1, \dots, \alpha_q\}$ where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq q-1$ and

$$(2.3) \quad \alpha_q = \begin{cases} \varepsilon_q & \text{if } p > q, \\ \varepsilon_{q-1} + \varepsilon_q & \text{if } p = q. \end{cases}$$

The opposition involution fixes the simple root $\alpha_1 = \varepsilon_1 - \varepsilon_2$. The parabolic subgroup $P_{\{\alpha_1\}} = P_{\{\alpha_1\}}^*$ will be denoted $P_1(b)$; it is the stabilizer in $O(b)$ of the line $\mathbf{R}e_1$. The opposite parabolic subgroup $P_{\{\alpha_1\}}^-$ is the stabilizer of $\mathbf{R}e_{p+q}$. The flag variety $\mathcal{F}_{\{\alpha_1\}} = O(b)/P_1(b)$ identifies with the space of b -isotropic lines in V (a closed subset of $\mathbf{P}(V)$) and will be denoted $\mathcal{F}_1(b)$. A pair (ℓ, ℓ') of elements of $\mathcal{F}_1(b)$ is transverse if and only if $\ell^{\perp b} + \ell' = V$.

Suppose $p > q$. For $1 \leq i \leq q$, the parabolic subgroup $P_{\{\alpha_i\}}$ will be denoted $P_i(b)$; it is the stabilizer in $O(b)$ of $\mathbf{R}e_1 \oplus \dots \oplus \mathbf{R}e_i$. It is conjugate to its opposite, $P_{\{\alpha_i\}}^-$, which is the stabilizer of $\mathbf{R}e_{p+q-i+1} \oplus \dots \oplus \mathbf{R}e_{p+q}$. The flag variety $\mathcal{F}_i(b) = O(b)/P_i(b)$ is the space of b -isotropic i -dimensional subspaces of V . A pair (W, W') in $\mathcal{F}_i(b)$ is transverse in the sense of Definition 2.1 if and only if $W^{\perp b} + W' = V$.

Suppose $p = q$. For $1 \leq i \leq p-1$ we denote again by $P_i(b)$ the stabilizer in $O(b)$ of the b -isotropic i -dimensional subspace $\mathbf{R}e_1 \oplus \dots \oplus \mathbf{R}e_i$. For $i < p-1$, $P_i(b)$ is $P_{\{\alpha_i\}}$ and $P_{p-1}(b)$ is $P_{\{\alpha_{p-1}, \alpha_p\}}$. For any $i < p$, $P_i(b)$ is conjugate to its opposite. The corresponding homogenous space $\mathcal{F}_i(b)$ is the space of b -isotropic i -planes of V . Transversality is as above. The parabolic subgroups $P_{\{\alpha_{p-1}\}}$ and $P_{\{\alpha_p\}}$ can be viewed as stabilizers of isotropic p -planes; they are always conjugate under an element of $O(b)$. The opposition involution fixes α_{p-1} and α_p if p is even and exchanges them if p is odd.

3. DIVERGENCE AND ANOSOV REPRESENTATIONS

In this section we recall the notion of limit set in a broad setting, as well as the definition of Anosov representations, and establish some properties of Anosov representations that will be used later in the paper. We continue with the notation of Section 2.

3.1. P_θ -divergence and limit sets. Let $\theta \subset \Delta$ be a nonempty subset of the simple restricted roots of the noncompact real reductive Lie group G . As in [GGKWa, § 5], we define a map $\Xi_\theta : G \rightarrow \mathcal{F}_\theta$ as follows: for any $g \in G$, we choose $k_g, \ell_g \in K$ such that $g = k_g \exp(\mu(g))\ell_g$, and set

$$(3.1) \quad \Xi_\theta(g) = k_g \cdot P_\theta \in \mathcal{F}_\theta = G/P_\theta.$$

This does not depend on the choice of k_g, ℓ_g as soon as $\langle \alpha, \mu(g) \rangle > 0$ for all $\alpha \in \theta$ (see [Hel01, Ch. IX, Cor. 1.2]). We adopt the following terminology.

Definition 3.1. A sequence $(g_n) \in G^{\mathbf{N}}$ is P_θ -divergent if for any $\alpha \in \theta$,

$$\lim_{n \rightarrow +\infty} \langle \alpha, \mu(g_n) \rangle = +\infty.$$

Definition 3.2. Let Λ be a discrete subgroup of G . Let $\theta \subset \Delta$ be a nonempty subset such that Λ admits a P_θ -divergent sequence. The *limit set* $\mathcal{L}_\Lambda^{\mathcal{F}_\theta}$ of Λ in \mathcal{F}_θ is the set of all limits in \mathcal{F}_θ of sequences $(\Xi_\theta(\gamma_n))_{n \in \mathbb{N}}$ where $(\gamma_n) \in \Lambda^\mathbb{N}$ is P_θ -divergent.

By [Ben97, § 3.2], if Λ is Zariski-dense in G , then Λ contains θ -proximal elements, i.e. elements with a unique attracting fixed point in \mathcal{F}_θ , and $\mathcal{L}_\Lambda^{\mathcal{F}_\theta}$ is the closure of the set of attracting fixed points of these elements.

Definition 3.3. Let Γ be a discrete group. A representation $\rho : \Gamma \rightarrow G$ is P_θ -divergent if all sequences of pairwise distinct elements in $\rho(\Gamma)$ are P_θ -divergent; equivalently, for any $\alpha \in \theta$, $\lim_{\gamma \rightarrow \infty} \langle \alpha, \mu(\rho(\gamma)) \rangle = +\infty$, i.e. for any $M > 0$ the set $\{\gamma \in \Gamma \mid \langle \alpha, \mu(\rho(\gamma)) \rangle < M\}$ is finite.

If $\rho : \Gamma \rightarrow G$ is P_θ -divergent, then it has finite kernel and discrete image.

- Remarks 3.4.* (1) A particular case of Definition 3.3 was used in [GW12, § 7.2]. The definition is equivalent to the notion of *weakly τ_{mod} -regular* subgroup of [KLPb, Def. 5.6] where τ_{mod} is the facet $\bar{\mathfrak{a}}^+ \cap \bigcap_{\alpha \in \Delta \setminus \theta} \text{Ker}(\alpha)$ of $\bar{\mathfrak{a}}^+$.
- (2) The equality $\langle \alpha, \mu(g) \rangle = \langle \alpha^*, \mu(g^{-1}) \rangle$ for all $\alpha \in \Delta$ and $g \in G$ implies that a representation $\rho : \Gamma \rightarrow G$ is P_θ -divergent if and only if it is $P_{\theta \cup \theta^*}$ -divergent.
- (3) If $g = k_g \exp(\mu(g)) \ell_g$ with $k_g, \ell_g \in K$, then $\Xi_\theta(g^{-1}) = \ell_g^{-1} \cdot P_{\theta^*}^-$ and $\Xi_\theta(g^{-1})$ does not depend on the choices as soon as $\langle \alpha, \mu(g) \rangle > 0$ for all $\alpha \in \theta^*$. Therefore, if a sequence $(\gamma_n)_{n \in \mathbb{N}}$ of $\Lambda^\mathbb{N}$ is P_{θ^*} -divergent and if the sequence $(\Xi_\theta(\gamma_n^{-1}))_{n \in \mathbb{N}}$ converges, then its limit belongs to the limit set $\mathcal{L}_\Lambda^{\mathcal{F}_\theta}$.

3.2. Anosov representations. We now suppose that Γ is word hyperbolic and denote by $\partial_\infty \Gamma$ its boundary at infinity. The following definition of Anosov representations is not the original one from [Lab06, GW12], but an equivalent one taken from [GGKWa].

Definition 3.5. Let Γ be a word hyperbolic group. A representation $\rho : \Gamma \rightarrow G$ is P_θ -Anosov if it is P_θ -divergent and there exist continuous, ρ -equivariant maps $\xi^+ : \partial_\infty \Gamma \rightarrow \mathcal{F}_\theta$ and $\xi^- : \partial_\infty \Gamma \rightarrow \mathcal{F}_{\theta^*}$ that are transverse and dynamics-preserving.

By *dynamics-preserving* we mean that if η is the attracting fixed point of some element $\gamma \in \Gamma$ in $\partial_\infty \Gamma$, then $\xi^+(\eta)$ (resp. $\xi^-(\eta)$) is an attracting fixed point of $\rho(\gamma)$ in \mathcal{F}_θ (resp. \mathcal{F}_{θ^*}). By *transverse* we mean that pairs of distinct points in $\partial_\infty \Gamma$ are sent to transverse pairs in $\mathcal{F}_\theta \times \mathcal{F}_{\theta^*}$ (Definition 2.1).

The maps ξ^+ and ξ^- are unique, entirely determined by ρ . The set of P_θ -Anosov representations is open in $\text{Hom}(\Gamma, G)$ [Lab06, GW12].

Remarks 3.6. (1) By Remark 3.4.(2) (see also [GW12, Lem. 3.18]), the representation $\rho : \Gamma \rightarrow G$ is P_θ -Anosov if and only if it is $P_{\theta \cup \theta^*}$ -Anosov.

(2) When $\theta = \theta^*$, the two flags varieties \mathcal{F}_θ and \mathcal{F}_{θ^*} coincide and the two boundary maps ξ^+ and ξ^- of a P_θ -Anosov representation are equal.

Example 3.7. Let $G = \text{GL}_\mathbf{R}(V)$ and $\theta = \{\alpha_i\} = \{\varepsilon_i - \varepsilon_{i+1}\}$. The boundary maps of a P_i -Anosov representation $\rho : \Gamma \rightarrow G$ are a pair of continuous maps

$$\xi_i = \xi^+ : \partial_\infty \Gamma \longrightarrow \text{Gr}_i(V) \quad \text{and} \quad \xi_{n-i} = \xi^- : \partial_\infty \Gamma \longrightarrow \text{Gr}_{n-i}(V)$$

such that $\xi_i(\eta) \oplus \xi_{n-i}(\eta') = V$ for all $\eta \neq \eta'$ in $\partial_\infty \Gamma$, and such that for any $\gamma \in \Gamma$ with attracting fixed point η in $\partial_\infty \Gamma$, the element $\rho(\gamma)$ has attracting fixed points $\xi_i(\eta)$ in $\text{Gr}_i(V)$ and $\xi_{n-i}(\eta)$ in $\text{Gr}_{n-i}(V)$. Here P_θ -divergence means

$$\lim_{\gamma \rightarrow \infty} \langle \varepsilon_i - \varepsilon_{i+1}, \mu(\rho(\gamma)) \rangle = +\infty.$$

Example 3.8. Let $G = \text{O}(b)$ be the orthogonal group of a symmetric bilinear form of signature (p, q) on a real vector space V , where $p, q \in \mathbf{N}^*$ and $(p, q) \neq (1, 1)$. Let $\theta = \{\alpha_i\} = \theta^*$ where $1 \leq i \leq \min(p, q)$; if $p = q$, we assume $i \neq p$ and $\theta = \{\alpha_{p-1}, \alpha_p\}$ when $i = p - 1$ (see (2.3)). By Remark 3.6.(2), for a $P_i(b)$ -Anosov representation $\rho : \Gamma \rightarrow G$ there is just one continuous ρ -equivariant boundary map $\xi : \partial_\infty \Gamma \rightarrow \mathcal{F}_i(b)$. It is dynamics-preserving and satisfies $\xi(\eta)^{\perp b} \oplus \xi(\eta') = V$ for all $\eta \neq \eta'$ in $\partial_\infty \Gamma$. Here P_θ -divergence means

$$\lim_{\gamma \rightarrow \infty} \langle \alpha_i, \mu(\rho(\gamma)) \rangle = +\infty.$$

If $p = q$ and $i = p - 1$, then the limit $\lim_{\gamma \rightarrow \infty} \langle \alpha_p, \mu(\rho(\gamma)) \rangle = +\infty$ is also part of P_θ -divergence.

For general G and θ , we shall use the following description of the limit set.

Lemma 3.9. [GGKWa, Th. 5.2] *If a representation $\rho : \Gamma \rightarrow G$ is P_θ -Anosov with boundary map $\xi^+ : \partial_\infty \Gamma \rightarrow \mathcal{F}_\theta$, then $\mathcal{L}_{\rho(\Gamma)}^{\mathcal{F}_\theta} = \xi^+(\partial_\infty \Gamma)$.*

3.3. θ -compatibility. We shall use the following terminology from [GGKWa].

Definition 3.10. Let V be a finite-dimensional real vector space and $\theta \subset \Delta$ a nonempty subset of the simple restricted roots of G . An irreducible representation $\tau : G \rightarrow \text{GL}_{\mathbf{R}}(V)$ with highest weight χ_τ is θ -compatible if

$$\{\alpha \in \Delta \mid (\chi_\tau, \alpha) > 0\} = \theta.$$

The following proposition was proved in [GGKWa] for $i = 1$.

Proposition 3.11. *Let (τ, V) be an irreducible, θ -compatible linear representation of G over \mathbf{R} . Let V^{χ_τ} be the weight space corresponding to the highest weight, let $i =: \dim_{\mathbf{R}}(V^{\chi_\tau}) < n =: \dim_{\mathbf{R}}(V)$, and let $V_{<\chi_\tau}$ be the sum of all the other weight spaces of τ .*

(1) *For any discrete group Γ and any representation $\rho : \Gamma \rightarrow G$, $\rho : \Gamma \rightarrow G$ is P_θ -divergent $\iff \tau \circ \rho : \Gamma \rightarrow \text{GL}_{\mathbf{R}}(V)$ is P_i -divergent.*

(2) *For any word hyperbolic group Γ and any representation $\rho : \Gamma \rightarrow G$, $\rho : \Gamma \rightarrow G$ is P_θ -Anosov $\iff \tau \circ \rho : \Gamma \rightarrow \text{GL}_{\mathbf{R}}(V)$ is P_i -Anosov.*

In this case, the boundary maps $\xi^+ : \partial_\infty \Gamma \rightarrow G/P_\theta$ and $\xi^- : \partial_\infty \Gamma \rightarrow G/P_\theta^-$ of ρ and the boundary maps $\xi_i : \partial_\infty \Gamma \rightarrow \text{Gr}_i(V)$ and $\xi_{n-i} : \partial_\infty \Gamma \rightarrow \text{Gr}_{n-i}(V)$ of $\tau \circ \rho$ are related as follows: for any $\eta \in \partial_\infty \Gamma$, if $(\xi^+(\eta), \xi^-(\eta)) = (gP_\theta, gP_\theta^-)$ where $g \in G$, then $(\xi_i(\eta), \xi_{n-i}(\eta)) = (\tau(g)V^{\chi_\tau}, \tau(g)V_{<\chi_\tau})$.

Proof. Identical to the proof of [GGKWa, Prop. 4.6 & 4.8]: one just needs to replace [GGKWa, Lem. 4.10.(3)] with the fact (following from [GGKWa, Lem. 4.10.(1)&(2)]) that for any $g \in G$,

$$\langle \alpha_i, \mu_{\text{GL}_{\mathbf{R}}(V)}(\tau(g)) \rangle = \min_{\alpha \in \theta} \langle \alpha, \mu_G(g) \rangle.$$

□

The following result is an easy consequence of Proposition 3.11 with $i = 1$; we shall use it to reduce to the group $O(b)$ in the proof of Theorem 1.1.

Proposition 3.12 ([GGKWa, Lem. 4.10, Prop. 6.7 & Rem. 6.9]). *Let $\theta \subset \Delta$ be a nonempty subset of the simple restricted roots of G . Then there exist a nondegenerate symmetric bilinear form b on a real vector space V and a homomorphism $\tau : G \rightarrow O(b)$ with the following properties:*

- (1) *For any discrete group Γ and any representation $\rho : \Gamma \rightarrow G$,
 $\rho : \Gamma \rightarrow G$ is P_θ -divergent $\iff \tau \circ \rho : \Gamma \rightarrow O(b)$ is $P_1(b)$ -divergent.*
- (2) *For any word hyperbolic group Γ and any representation $\rho : \Gamma \rightarrow G$,
 $\rho : \Gamma \rightarrow G$ is P_θ -Anosov $\iff \tau \circ \rho : \Gamma \rightarrow O(b)$ is $P_1(b)$ -Anosov.*

There are infinitely many such triples (p, q, τ) , see [GGKWa].

Lemma 3.13. *Let b be a nondegenerate symmetric bilinear form of signature (p, q) on a real vector space V , where $p, q \in \mathbf{N}^*$ and $(p, q) \neq (1, 1)$. Let $1 \leq i \leq \min(p, q)$, with $i < p$ if $p = q$. Let $\iota : O(b) \hookrightarrow GL_{\mathbf{R}}(V)$ be the natural inclusion.*

- (1) *For any discrete group Γ and any representation $\rho : \Gamma \rightarrow O(b)$,
 $\rho : \Gamma \rightarrow O(b)$ is $P_i(b)$ -divergent $\iff \tau \circ \rho : \Gamma \rightarrow GL_{\mathbf{R}}(V)$ is P_i -divergent.*
- (2) *For any word hyperbolic group Γ and any representation $\rho : \Gamma \rightarrow O(b)$,
 $\rho : \Gamma \rightarrow O(b)$ is $P_i(b)$ -Anosov $\iff \tau \circ \rho : \Gamma \rightarrow GL_{\mathbf{R}}(V)$ is P_i -Anosov.*

Proof. The action of $O(b)$ on the exterior product $\bigwedge^i V$ is irreducible and α_i -compatible, and the highest weight space has dimension 1. By Proposition 3.11, the representation ρ is $P_i(b)$ -divergent (resp. $P_i(b)$ -Anosov) if and only if $\bigwedge^i \rho : \Gamma \rightarrow GL_{\mathbf{R}}(\bigwedge^i V)$ is P_1 -divergent (resp. P_1 -Anosov). The same proposition, applied to the linear representation $GL_{\mathbf{R}}(V) \rightarrow GL_{\mathbf{R}}(\bigwedge^i V)$, implies that $\iota \circ \rho$ is P_i -divergent (resp. P_i -Anosov) if and only if $\bigwedge^i \rho : \Gamma \rightarrow GL_{\mathbf{R}}(\bigwedge^i V)$ is P_1 -divergent (resp. P_1 -Anosov). The lemma follows. \square

3.4. The adjoint representation. For a noncompact semisimple Lie group G , recall that the Killing form κ of the Lie algebra \mathfrak{g} is a nondegenerate indefinite symmetric bilinear form on \mathfrak{g} . Let $\text{Ad} : G \rightarrow O(\kappa) \subset GL_{\mathbf{R}}(\mathfrak{g})$ be the adjoint representation. The highest restricted weight $\chi_G \in \Sigma^+$ of Ad is called the *highest restricted root*. In the case that G is simple, we prove the following.

Proposition 3.14. *Let G be a real simple Lie group.*

- (1) *There exists a simple restricted root $\alpha_G \in \Delta$ such that $\text{Ad} : G \rightarrow GL_{\mathbf{R}}(\mathfrak{g})$ is $\{\alpha_G, \alpha_G^*\}$ -compatible (Definition 3.10), i.e.*

$$\{\alpha \in \Delta \mid (\chi_G, \alpha) > 0\} = \{\alpha_G, \alpha_G^*\}$$

where $\chi_G \in \Sigma^+$ is the highest restricted root. Moreover, $\alpha_G = \alpha_G^*$ unless the restricted root system Σ is of type A_n .

Let d be the real dimension of the root space \mathfrak{g}_{χ_G} .

- (2) *For any discrete group Γ and any representation $\rho : \Gamma \rightarrow G$,
 $\rho : \Gamma \rightarrow G$ is $P_{\{\alpha_G\}}$ -divergent $\iff \text{Ad} \circ \rho : \Gamma \rightarrow O(\kappa)$ is $P_d(\kappa)$ -divergent.*
- (3) *For any word hyperbolic group Γ and any representation $\rho : \Gamma \rightarrow G$,
 $\rho : \Gamma \rightarrow G$ is $P_{\{\alpha_G\}}$ -Anosov $\iff \text{Ad} \circ \rho : \Gamma \rightarrow O(\kappa)$ is $P_d(\kappa)$ -Anosov.*

In this case the boundary map $\xi : \partial_\infty \Gamma \rightarrow G/P_{\{\alpha_G, \alpha_G^*\}}$ of ρ and the boundary map $\xi_d : \partial_\infty \Gamma \rightarrow \mathcal{F}_d(\kappa)$ of $\text{Ad} \circ \rho$ are related as follows: for any $\eta \in \partial_\infty \Gamma$, if $\xi(\eta) = g \cdot P_{\{\alpha_G, \alpha_G^*\}}$ where $g \in G$, then $\xi_d(\eta) = \text{Ad}(g) \cdot \mathfrak{g}_{\chi_G}$.

Table 1 gives the simple root α_G and the highest weight χ_G for the various restricted root systems, see [Hel01, Ch. X, Th. 3.28].

Type	α_G	χ_G
A_n	α_1	$\varepsilon_1 - \varepsilon_{n+1} = \alpha_1 + \cdots + \alpha_n$
B_n	α_2	$\varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$
C_n	α_1	$2\varepsilon_1 = 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n$
BC_n	α_1	$2\varepsilon_1 = 2\alpha_1 + \cdots + 2\alpha_n$
D_n	α_2	$\varepsilon_1 + \varepsilon_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$
E_6	α_4	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7	α_6	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8	α_7	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$
F_4	α_1	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
G_2	α_1	$3\alpha_1 + 2\alpha_2$

TABLE 1. The simple root α_G and the highest root χ_G according to the Dynkin diagram of the restricted root system

Proof. The set $\{\alpha \in \Delta \mid (\chi_G, \alpha) > 0\}$ is the set of simple roots connected to the added node in the extended Dynkin diagram. The result is thus a consequence of the classification of those diagrams, see e.g. [Bou68, Ch. VI, § 4, no. 3]. Since Ad is selfdual, $\chi_G^* = \chi_G$ and $\langle \alpha, \chi_G \rangle \neq 0$ if and only if $\langle \alpha^*, \chi_G \rangle \neq 0$. This proves (1).

Let Γ be a discrete (resp. word hyperbolic) group and $\rho : \Gamma \rightarrow G$ a representation. Proposition 3.11 implies that ρ is $P_{\{\alpha_G\}}$ -divergent (resp. $P_{\{\alpha_G\}}$ -Anosov) if and only if $\text{Ad} \circ \rho : \Gamma \rightarrow \text{GL}_{\mathbf{R}}(\mathfrak{g})$ is P_d -divergent (resp. P_d -Anosov). On the other hand, Lemma 3.13 implies that $\text{Ad} \circ \rho : \Gamma \rightarrow \text{O}(\kappa)$ is $P_d(\kappa)$ -divergent (resp. $P_d(\kappa)$ -Anosov) if and only if $\text{Ad} \circ \rho : \Gamma \rightarrow \text{GL}_{\mathbf{R}}(\mathfrak{g})$ is P_d -divergent (resp. P_d -Anosov). This proves (2) and (3). \square

4. COMPACTIFYING RIEMANNIAN LOCALLY SYMMETRIC SPACES: THE CASE OF ORTHOGONAL GROUPS

In this section we construct a compactification for Riemannian locally symmetric spaces arising from $P_1(b)$ -Anosov representations into $\text{O}(b)$.

For a nondegenerate symmetric bilinear form b of signature (p, q) on a real vector space V , the Riemannian symmetric space X_b of $\text{O}(b)$ admits a realization as an open subset in the Grassmannian $\text{Gr}_q(V)$, namely as the set of $W \in \text{Gr}_q(V)$ such that the restriction of b to $W \times W$ is negative definite. Its closure

$$(4.1) \quad \overline{X}_b = \{W \in \text{Gr}_q(V) \mid b(x, x) \leq 0 \quad \forall x \in W\}$$

in $\text{Gr}_q(V)$ is a compactification of X_b . This compactification is isomorphic to a minimal Satake compactification of X_b if $p > q$ (see Section A.3), and a generalized Satake compactification if $p = q$ (see Section A.4). The main result that we prove in this section is the following.

Theorem 4.1. *Let b be a nondegenerate symmetric bilinear form of signature (p, q) on a real vector space V , where $p, q \in \mathbf{N}^*$ and $(p, q) \neq (1, 1), (2, 2)$. Let Γ be a word hyperbolic group and $\rho : \Gamma \rightarrow \mathrm{O}(b)$ a $P_1(b)$ -Anosov representation with boundary map $\xi : \partial_\infty \Gamma \rightarrow \mathcal{F}_1(b)$. Let*

$$\mathcal{N}_\rho = \bigcup_{\eta \in \partial_\infty \Gamma} \{W \in \bar{X}_b \mid \xi(\eta) \subset W\}.$$

Then the action of Γ via ρ on $\Omega = \bar{X}_b \setminus \mathcal{N}_\rho$ is properly discontinuous and cocompact. The set Ω contains the Riemannian symmetric space X_b and $\rho(\Gamma) \backslash \Omega$ is a compactification of $\rho(\Gamma) \backslash X_b$.

Properness will be proved in Section 4.2 and cocompactness in Section 4.3.

4.1. Nonpositive quadratic spaces. We establish the following elementary lemma, used later in the proof of Theorem 4.1. We denote by

$$\mathrm{Ker}(b) = \{y \in V \mid b(x, y) = 0 \ \forall x \in V\}$$

the kernel of a symmetric bilinear form b on a real vector space V .

Lemma 4.2. *Let b be a nondegenerate symmetric bilinear form of signature (p, q) on a real vector space V . Let $W \in \mathrm{Gr}_q(V)$ satisfy $b(x, x) \leq 0$ for all $x \in W$.*

- (1) *If $y \in W$ satisfies $b(y, y) = 0$, then $y \in \mathrm{Ker}(b|_{W \times W})$. Conversely, if $y \in V$ satisfies $b(y, y) = 0$ and $b(x, y) = 0$ for all $x \in W$, then $y \in W$.*
- (2) *For any b -isotropic subspace L of V ,*

$$\begin{aligned} L \cap W \neq \{0\} &\iff W + L^{\perp_b} \neq V, \text{ and} \\ L \subset W &\iff W \subset L^{\perp_b}. \end{aligned}$$

Proof of Lemma 4.2. (1) If $y \in W$ satisfies $b(y, y) = 0$, then $b(x, y) = 0$ for all $x \in W$, otherwise we would have $b(x + ty, x + ty) = b(x, x) + 2tb(x, y) > 0$ for certain values of $t \in \mathbf{R}$.

Conversely, let $y \in V$ satisfy $b(y, y) = 0$ and $b(x, y) = 0$ for all $x \in W$. Let $\ell = \mathbf{R}y \subset V$. The projection Z of W to ℓ^{\perp_b}/ℓ is a nonpositive subspace in a vector space equipped with a nondegenerate symmetric bilinear form of signature $(p - 1, q - 1)$. Such a subspace Z has trivial intersection with any $(p - 1)$ -dimensional positive subspace, hence $\dim_{\mathbf{R}}(Z) \leq q - 1$. Since $\dim_{\mathbf{R}}(W) = q$, we deduce $\ell \subset W$ and $y \in W$.

(2) Suppose $L \cap W \neq \{0\}$ and let y be non zero in $L \cap W$. Then $W \subset y^{\perp_b}$ by (1). As L is isotropic, $L^{\perp_b} \subset y^{\perp_b}$, and one gets $W + L^{\perp_b} \subset y^{\perp_b}$ and $W + L^{\perp_b} \neq V$.

Conversely, suppose $W + L^{\perp_b} \neq V$. Let $H \subset V$ be a hyperplane containing W and L^{\perp_b} and $y \in V$ such that $y^{\perp_b} = H$. By duality $y \in L$ and y is isotropic. By (1) we have $y \in W$, hence $y \in W \cap L$ and $W \cap L \neq \{0\}$. The equivalence $L \subset W \iff W \subset L^{\perp_b}$ follows from (1) as well. \square

4.2. Proper discontinuity. Properness in Theorem 4.1 is an immediate consequence of Lemma 3.9 and of the following proposition with $i = 1$. We refer to Definitions 3.2 and 3.3 for the notions of limit set and divergent representation, and to Section 2.5 and Example 3.8 for the assumptions on i .

Proposition 4.3. *Let b be a nondegenerate symmetric bilinear form of signature (p, q) on a real vector space V , where $p, q \in \mathbf{N}^*$ and $(p, q) \neq (1, 1)$. Let $1 \leq i \leq \min(p, q)$; if $p = q$, assume that $i < p$. Let Γ be a discrete group and $\rho : \Gamma \rightarrow \mathrm{O}(b)$ a $P_i(b)$ -divergent representation. Let*

$$\mathcal{W}_\rho^i = \bigcup_{L \in \mathcal{L}_{\rho(\Gamma)}^{\mathcal{F}_i(b)}} \{W \in \bar{X}_b \mid L \cap W \neq \{0\}\},$$

where $\mathcal{L}_{\rho(\Gamma)}^{\mathcal{F}_i(b)} \subset \mathcal{F}_i(b)$ is the limit set of $\rho(\Gamma)$. Then $\bar{X}_b \setminus \mathcal{W}_\rho^i$ contains X_b and the action of Γ on $\bar{X}_b \setminus \mathcal{W}_\rho^i$ is properly discontinuous.

In fact we prove the following very general statement.

Proposition 4.4. *Let b be a nondegenerate symmetric bilinear form of signature (p, q) on a real vector space V , where $p, q \in \mathbf{N}^*$ and $p \neq q$. Let Γ be a discrete group and $\rho : \Gamma \rightarrow \mathrm{O}(b)$ any representation with discrete image and finite kernel. Let*

$$\mathcal{W}_\rho = \bigcup_{i \in I_\rho} \bigcup_{L \in \mathcal{L}_{\rho(\Gamma)}^{\mathcal{F}_i(b)}} \{W \in \bar{X}_b \mid L \cap W \neq \{0\}\},$$

where $I_\rho \subset \{1, \dots, \min(p, q)\}$ is the set of integers i such that $\langle \alpha_i, \mu(\rho(\Gamma)) \rangle$ is unbounded, and $\mathcal{L}_{\rho(\Gamma)}^{\mathcal{F}_i(b)} \subset \mathcal{F}_i(b)$ is the limit set of $\rho(\Gamma)$. Then $\bar{X}_b \setminus \mathcal{W}_\rho$ contains X_b and the action of Γ on $\bar{X}_b \setminus \mathcal{W}_\rho$ is properly discontinuous.

Remarks 4.5. (1) Proposition 4.4 provides a bordification of $\Lambda \backslash X_b$ locally modeled on \bar{X}_b for any discrete subgroup Λ of $\mathrm{O}(b)$. From this we deduce a bordification of $\Lambda \backslash G/K$ as a manifold with corners for any discrete subgroup Λ of any semisimple Lie group G (Theorem 5.4).

(2) In [KL], bordifications are constructed, by a different method, for discrete subgroups Γ of a simple group G that are *uniformly τ_{mod} -regular* for some facet τ_{mod} of $\bar{\mathfrak{a}}^+$. If we write $\tau_{\mathrm{mod}} = \bar{\mathfrak{a}}^+ \cap \bigcap_{\alpha \in \Delta \setminus \theta} \mathrm{Ker}(\alpha)$ for some nonempty $\theta \subset \Delta$, then these are the discrete subgroups Γ of G for which there exist $c, C > 0$ such that for any $\alpha \in \theta$ and any $\gamma \in \Gamma$,

$$\langle \alpha, \mu(\gamma) \rangle \geq c \|\mu(\gamma)\| - C,$$

where $\|\cdot\|$ is a fixed norm on \mathfrak{a} . In other words, Γ is P_θ -divergent with a linear rate of divergence.

Recall that two points x and x' of X are said to be *dynamically related* if there exist a sequence $(x_n)_{n \in \mathbf{N}}$ in $X^\mathbf{N}$ converging to x and a sequence $(\gamma_n)_{n \in \mathbf{N}} \in \Gamma^\mathbf{N}$ going to infinity (i.e. leaving every finite subset of Γ) such that the sequence $(\gamma_n \cdot x_n)_{n \in \mathbf{N}}$ converges to x' . Propositions 4.3 and 4.4 are immediate consequences of the following classical dynamical criterion for properness (see e.g. [Fra05] for a proof) and of the following lemma.

Criterion 4.6. A group Γ acts properly discontinuously on a Hausdorff topological space X if and only if no pairs of points of X are dynamically related.

Lemma 4.7. *In the setting of Proposition 4.3, consider an arbitrary sequence $(W_n)_{n \in \mathbf{N}} \in (\bar{X}_b \setminus \mathcal{W}_\rho^i)^{\mathbf{N}}$ converging to some $W \in \bar{X}_b \setminus \mathcal{W}_\rho^i$ and an arbitrary $P_i(b)$ -divergent sequence $(\rho(\gamma_n))_{n \in \mathbf{N}} \in \Gamma^{\mathbf{N}}$ such that $(W'_n = \rho(\gamma_n) \cdot W_n)_{n \in \mathbf{N}}$ converges to some $W' \in \text{Gr}_q(V)$. Then $W' \in \mathcal{W}_\rho^i$.*

Proof of Lemma 4.7. We write $\rho(\gamma_n) = k_n a_n \ell_n \in K \exp(\bar{\mathfrak{a}}^+) K$. Up to extracting, we can assume that the sequences $(k_n)_{n \in \mathbf{N}}$ and $(\ell_n)_{n \in \mathbf{N}}$ converge to some $k_\infty, \ell_\infty \in K$, respectively. By Definition 3.2 of the limit set (see Section 2.5 and Remark 3.4.(3)),

$$L^+ := k_\infty \cdot (\mathbf{R}e_1 \oplus \cdots \oplus \mathbf{R}e_i) \quad \text{and} \quad L^- := \ell_\infty^{-1} \cdot (\mathbf{R}e_{p+q-i+1} \oplus \cdots \oplus \mathbf{R}e_{p+q})$$

belong to the limit set $\mathcal{L}_{\rho(\Gamma)}^{\mathcal{F}_i(b)}$. The assumption $W \cap L^- = \{0\}$ and Lemma 4.2 imply that W is not contained in $(L^-)^{\perp_b} = \ell_\infty^{-1} \cdot (\mathbf{R}e_{i+1} \oplus \cdots \oplus \mathbf{R}e_{p+q})$. This means that there exist $w_\infty \in W$ and $c_1, \dots, c_{p+q} \in \mathbf{R}$ such that

$$\ell_\infty \cdot w_\infty = \sum_{j=1}^{p+q} c_j e_j,$$

and $(c_1, \dots, c_i) \neq 0$. There is a sequence $(w_n)_{n \in \mathbf{N}} \in V^{\mathbf{N}}$ converging to w_∞ such that $w_n \in W_n$ for all n . The sequence $(\ell_n \cdot w_n)_{n \in \mathbf{N}}$ converges to $\ell_\infty \cdot w_\infty$. We write $\ell_n \cdot w_n = \sum_{j=1}^{p+q} c_{j,n} e_j$, thus $\lim_n c_{j,n} = c_j$ for any $j \in \{1, \dots, p+q\}$.

For $n \in \mathbf{N}$, let r_n be the inverse of the Euclidean norm of the vector $(e^{\langle \varepsilon_j, \log a_n \rangle} c_{j,n})_{j=1, \dots, i} \in \mathbf{R}^i$. Set $d_{j,n} := r_n e^{\langle \varepsilon_j, \log a_n \rangle} c_{j,n}$, for $j \in \{1, \dots, i\}$ and $n \in \mathbf{N}$. Up to extracting a subsequence, the sequence of i -tuples $(d_{1,n}, \dots, d_{i,n})_{n \in \mathbf{N}}$ converges to some (d_1, \dots, d_i) of norm 1 in \mathbf{R}^i . Consider $j_0 \in \{1, \dots, i\}$ such that $c_{j_0} \neq 0$, the sequence

$$(r_n e^{\langle \varepsilon_{j_0}, \log a_n \rangle})_{n \in \mathbf{N}} = \left(\frac{d_{j_0,n}}{c_{j_0,n}} \right)_{n \in \mathbf{N}}$$

converges to d_{j_0}/c_{j_0} and is thus bounded. This implies that for every $j > i$ the sequence $(r_n e^{\langle \varepsilon_j, \log a_n \rangle} c_{j,n})_{n \in \mathbf{N}}$ converges to zero since

$$\begin{aligned} r_n e^{\langle \varepsilon_j, \log a_n \rangle} c_{j,n} &= r_n e^{\langle \varepsilon_{j_0}, \log a_n \rangle} c_{j,n} e^{-\langle \varepsilon_{j_0} - \varepsilon_j, \log a_n \rangle} \\ &\leq r_n e^{\langle \varepsilon_{j_0}, \log a_n \rangle} c_{j,n} e^{-\langle \alpha_i, \log a_n \rangle}, \end{aligned}$$

which converges to 0 by P_i -divergence of $(\rho(\gamma_n))_{n \in \mathbf{N}}$.

We claim that the sequence $(v_n)_{n \in \mathbf{N}}$ defined by

$$v_n = r_n \rho(\gamma_n) \cdot w_n \in W'_n, \quad \text{for all } n \in \mathbf{N},$$

converges to $v_\infty = k_\infty \cdot (d_1 e_1 + \cdots + d_i e_i) \in L^+$. Indeed,

$$\begin{aligned} k_n^{-1} \cdot v_n &= r_n a_n \ell_n \cdot w_n = \sum_{j=1}^{p+q} r_n c_{j,n} a_n \cdot e_j \\ &= \sum_{j=1}^{p+q} r_n c_{j,n} e^{\langle \varepsilon_j, \log a_n \rangle} e_j \xrightarrow{n \rightarrow \infty} \sum_{j=1}^i d_j e_j. \end{aligned}$$

By the convergence of $(W'_n)_{n \in \mathbf{N}}$ to W' , v_∞ belongs to W' as well. Hence W' has nontrivial intersection with L^+ and belongs to \mathcal{W}_ρ^i . \square

Proof of Proposition 4.4. Apply Criterion 4.6 and Lemma 4.7, using the fact that if $\rho : \Gamma \rightarrow G$ is discrete with finite kernel, then for any sequence $(\gamma_n)_{n \in \mathbb{N}} \in \Gamma^{\mathbb{N}}$ going to infinity, up to passing to a subsequence, there exists i such that $(\rho(\gamma_n))_{n \in \mathbb{N}}$ is $P_{\{\alpha_i\}}$ -divergent (by properness of the map μ). \square

4.3. Compactness. We now restrict to a special class of divergent representations, namely Anosov representations (Definition 3.5). Compactness of $\rho(\Gamma) \backslash \Omega$ in Theorem 4.1 is a consequence of the following general result.

Lemma 4.8. *Let b be a nondegenerate symmetric bilinear form of signature (p, q) on a real vector space V , where $p, q \in \mathbb{N}^*$ and $(p, q) \neq (1, 1)$. Let $1 \leq i \leq \min(p, q)$; if $p = q$, assume that $i < p - 1$. For any word hyperbolic group Γ and any $P_i(b)$ -Anosov representation $\rho : \Gamma \rightarrow \mathrm{O}(b)$ with boundary map $\xi_i : \partial_\infty \Gamma \rightarrow \mathcal{F}_i(b)$, let*

$$\mathcal{V}_\rho = \bigcup_{\eta \in \partial_\infty \Gamma} \{W \in \bar{X}_b \mid \xi_i(\eta) \subset W\}.$$

Then the action of Γ on $\bar{X}_b \setminus \mathcal{V}_\rho$ is cocompact.

Lemma 4.8 itself is a direct consequence of Lemma 3.13.(2), of the fact that $\bar{X}_b \subset \mathrm{Gr}_q(V)$ is closed, and of the following statement.

Lemma 4.9. *Let V be a real vector space of dimension n , let $1 \leq i \leq q \leq n - 1$ be two integers, and let $\rho : \Gamma \rightarrow \mathrm{GL}_{\mathbf{R}}(V)$ be a P_i -Anosov representation with boundary maps $\xi_i : \partial_\infty \Gamma \rightarrow \mathrm{Gr}_i(V)$ and $\xi_{n-i} : \partial_\infty \Gamma \rightarrow \mathrm{Gr}_{n-i}(V)$. Let*

$$\mathcal{B}_\rho = \bigcup_{\eta \in \partial_\infty \Gamma} \{W \in \mathrm{Gr}_q(V) \mid \xi_i(\eta) \subset W\}.$$

Then the action of Γ on $\mathrm{Gr}_q(V) \setminus \mathcal{B}_\rho$ is cocompact.

The rest of this section is devoted to proving Lemma 4.9. Let us first introduce some notation. In a metric space (X, d_X) the distance from a point to a set will be denoted by dist_X and the Hausdorff distances by Hdist_X . For lines $L, L' \in \mathbf{P}(V)$ with respective direction vectors v and v' , we set

$$d_{\mathbf{P}(V)}(L, L') := |\sin \angle(v, v')|.$$

For $W, W' \in \mathrm{Gr}_q(V)$, we set

$$d_{\mathrm{Gr}_q(V)}(W, W') = \mathrm{Hdist}_{\mathbf{P}(V)}(\mathbf{P}(W), \mathbf{P}(W')).$$

For $L \in \mathbf{P}(V)$, we set

$$\mathcal{K}_L := \{W \in \mathrm{Gr}_q(V) \mid L \subset W\}.$$

Note that $\mathcal{K}_{g \cdot L} = g \cdot \mathcal{K}_L$ for all $g \in \mathrm{GL}_{\mathbf{R}}(V)$. The following identity is easily established:

$$(4.2) \quad \mathrm{dist}_{\mathrm{Gr}_q(V)}(W, \mathcal{K}_L) = \mathrm{dist}_{\mathbf{P}(V)}(L, \mathbf{P}(W)).$$

The following result is a consequence of estimates established in [GGKWa].

Proposition 4.10. *Let V be a real vector space of dimension n , let $\rho : \Gamma \rightarrow \mathrm{GL}_{\mathbf{R}}(V)$ be a P_1 -Anosov representation with boundary map $\xi_1 : \partial_\infty \Gamma \rightarrow \mathbf{P}(V)$, and let $1 \leq q \leq n - 1$. Then for any $\eta \in \partial_\infty \Gamma$ and any $c > 1$, there exist $\gamma \in \Gamma$ and an open subset U of $\mathrm{Gr}_q(V)$ containing $\mathcal{K}_{\xi_1(\eta)}$ such that*

$$\mathrm{dist}_{\mathrm{Gr}_q(V)}(\rho(\gamma) \cdot W, \rho(\gamma) \cdot \mathcal{K}_L) \geq c \mathrm{dist}_{\mathrm{Gr}_q(V)}(W, \mathcal{K}_L)$$

for all $W \in U$ and $L \in \mathbf{P}(V)$ with $\mathcal{K}_L \subset U$.

Proof of Proposition 4.10. Let (e_1, \dots, e_n) be a basis of V so that $G = \mathrm{GL}_{\mathbf{R}}(V)$ admits a Cartan decomposition $G = K \exp(\bar{\mathfrak{a}}^+)K$ as in Example 2.4. Fix $\eta \in \partial_{\infty}\Gamma$ and let $(\gamma_n)_{n \in \mathbf{N}}$ be a quasigeodesic ray in Γ converging to η . For any $n \in \mathbf{N}$ we write $\rho(\gamma_n) = k_n a_n \ell_n \in K \exp(\bar{\mathfrak{a}}^+)K$. Let $x_0 = \mathbf{R}e_1 \in \mathbf{P}(V)$. By [GGKWa, Th. 1.3.(1) \Rightarrow (4) & Th. 5.1.(1)],

$$\xi_1(\eta) = \lim_{n \rightarrow \infty} k_n \cdot x_0.$$

By [GGKWa, §5.4.1] (see the proof of Prop. 5.11 of [GGKWa], establishing (5.8) in Lem. 5.12), there exist $\delta > 0$ and $N \in \mathbf{N}$ such that for all $n \geq N$,

$$\limsup_{m \rightarrow \infty} d_{\mathbf{P}(V)}(x_0, (a_n^{-1} k_n^{-1} k_{n+m} a_n) \cdot x_0) \leq 1 - \delta.$$

Since $a_n \cdot x_0 = x_0$ and since the metric $d_{\mathbf{P}(V)}$ is invariant under $\ell_n^{-1} \in K$, we deduce that

$$d_{\mathbf{P}(V)}(\ell_n^{-1} \cdot x_0, \rho(\gamma_n)^{-1} \cdot \xi_1(\eta)) \leq 1 - \delta$$

for all $n \geq N$. We set

$$\mathcal{U}_n = \left\{ W \in \mathrm{Gr}_q(V) \mid \mathrm{dist}_{\mathbf{P}(V)}(\ell_n^{-1} \cdot x_0, \mathbf{P}(W)) < 1 - \frac{\delta}{2} \right\}.$$

By (4.2), for any $L \in \mathbf{P}(V)$ we have

$$(4.3) \quad \mathcal{K}_L \subset \mathcal{U}_n \iff d_{\mathbf{P}(V)}(\ell_n^{-1} \cdot x_0, L) < 1 - \frac{\delta}{2}.$$

Therefore, $\mathcal{K}_{\rho(\gamma_n)^{-1} \cdot \xi_1(\eta)} \subset \mathcal{U}_n$ for all $n \geq N$. To conclude the proof, it is enough to establish the following.

Claim 4.11. *For any $c > 1$, there exists $n_c \in \mathbf{N}$ such that for all $n \geq n_c$, all $W \in \mathcal{U}_n$, and all $L \in \mathbf{P}(V)$ with $\mathcal{K}_L \subset \mathcal{U}_n$,*

$$(4.4) \quad \mathrm{dist}_{\mathrm{Gr}_q(V)}(W, \mathcal{K}_L) \geq c \mathrm{dist}_{\mathrm{Gr}_q(V)}(\rho(\gamma_n) \cdot W, \rho(\gamma_n) \cdot \mathcal{K}_L).$$

Indeed, Proposition 4.10 follows from Claim 4.11 by setting $\gamma = \gamma_n^{-1}$ and $U = \rho(\gamma_n) \cdot \mathcal{U}_n$ for $n \geq \max(N, n_c)$.

We now prove Claim 4.11. It is a consequence of the following two elementary estimates:

- (a) There exist $\epsilon, M > 0$ such that for any $L' \in \mathbf{P}(V)$ and $W' \in \mathrm{Gr}_q(V)$, if $d_{\mathbf{P}(V)}(x_0, L') < 1 - \delta/2$, then

$$\mathrm{dist}_{\mathbf{P}(V)}(L', \mathbf{P}(W') \cap \overline{B}_{\mathbf{P}(V)}(x_0, 1 - \epsilon)) \leq M \mathrm{dist}_{\mathbf{P}(V)}(L', \mathbf{P}(W')).$$

- (b) For any $\epsilon > 0$ and $c' > 0$, there exists $n_{\epsilon, c'}$ such that $a_n|_{\overline{B}_{\mathbf{P}(V)}(x_0, 1 - \epsilon)}$ is c' -contracting for all $n \geq n_{\epsilon, c'}$.

Indeed, fix $c > 1$. For $n \geq N$, consider $W \in \mathcal{U}_n$ and $L \in \mathbf{P}(V)$ with $\mathcal{K}_L \subset \mathcal{U}_n$. Set $W' = \ell_n \cdot W$ and $L' = \ell_n \cdot L$. By (4.3), we have $d_{\mathbf{P}(V)}(x_0, L') < 1 - \frac{\delta}{2}$. By (a) and (b), if $n \geq n_{\epsilon, c/M}$, then

$$\begin{aligned} \mathrm{dist}_{\mathbf{P}(V)}(a_n \cdot L', a_n \cdot \mathbf{P}(W')) &\leq \mathrm{dist}_{\mathbf{P}(V)}(a_n \cdot L', a_n \cdot (\mathbf{P}(W') \cap \overline{B}_{\mathbf{P}(V)}(x_0, 1 - \epsilon))) \\ &\leq \frac{c}{M} \mathrm{dist}_{\mathbf{P}(V)}(L', (\mathbf{P}(W') \cap \overline{B}_{\mathbf{P}(V)}(x_0, 1 - \epsilon))) \\ &\leq c \mathrm{dist}_{\mathbf{P}(V)}(L', \mathbf{P}(W')). \end{aligned}$$

On the other hand, by (4.2) and the fact that the metric $d_{\text{Gr}_q(V)}$ is invariant under K ,

$$\begin{cases} \text{dist}_{\text{Gr}_q(V)}(W, \mathcal{K}_L) = \text{dist}_{\mathbf{P}(V)}(L', \mathbf{P}(W')), \\ \text{dist}_{\text{Gr}_q(V)}(\rho(\gamma_n) \cdot W, \rho(\gamma_n) \cdot \mathcal{K}_L) = \text{dist}_{\mathbf{P}(V)}(a_n \cdot L', a_n \cdot \mathbf{P}(W')). \end{cases}$$

This concludes the proof of Claim 4.11.

For the sake of completeness, we now give a proof of (a) and (b) above: (b) is a consequence of the fact that $\langle \varepsilon_1 - \varepsilon_2, \log a_n \rangle \rightarrow +\infty$ (since ρ is P_1 -divergent by Definition 3.5 of a P_1 -Anosov representation). For (a) we argue by contradiction: suppose that there are sequences $(\epsilon_m)_{m \in \mathbf{N}} \in (\mathbf{R}_{>0})^{\mathbf{N}}$, $(M_m)_{m \in \mathbf{N}} \in (\mathbf{R}_{>0})^{\mathbf{N}}$, $(L'_m)_{m \in \mathbf{N}} \in \mathbf{P}(V)^{\mathbf{N}}$, and $(W'_m)_{m \in \mathbf{N}} \in \text{Gr}_q(V)^{\mathbf{N}}$ such that $(\epsilon_m)_{m \in \mathbf{N}}$ converges to 0, such that $(M_m)_{m \in \mathbf{N}}$ diverges to $+\infty$, and such that for any m we have $d_{\mathbf{P}(V)}(x_0, L'_m) < 1 - \delta/2$ and

$$(4.5) \quad \text{dist}_{\mathbf{P}(V)}(L'_m, \mathbf{P}(W'_m) \cap \overline{B}_{\mathbf{P}(V)}(x_0, 1 - \epsilon_m)) > M_m \text{dist}_{\mathbf{P}(V)}(L'_m, \mathbf{P}(W'_m)).$$

For any m , the left-hand side of (4.5) is ≤ 1 , hence $\text{dist}_{\mathbf{P}(V)}(L'_m, \mathbf{P}(W'_m)) \leq 1/M_m$. Let $D_m \subset W'_m$ be a line such that $d_{\mathbf{P}(V)}(L'_m, D_m) = \text{dist}_{\mathbf{P}(V)}(L'_m, \mathbf{P}(W'_m))$. Then $d_{\mathbf{P}(V)}(x_0, D_m) \leq d_{\mathbf{P}(V)}(x_0, L'_m) + d_{\mathbf{P}(V)}(L'_m, D_m) \leq 1 - \delta/2 + 1/M_m$, and so D_m belongs to $\overline{B}_{\mathbf{P}(V)}(x_0, 1 - \epsilon_m)$ for large enough m , contradicting (4.5). \square

Lemma 4.9 is a consequence of Proposition 4.10 and of the following dynamical compactness criterion from [KLPa], inspired by Sullivan's dynamical characterization of convex cocompactness [Sul85]. We recall the proof for the reader's convenience.

Lemma 4.12 ([KLPa, Prop. 2.5]). *Let Λ be a group acting by homeomorphisms on a compact metric space (Z, d_Z) and on a compact set D . Let E be a closed Λ -invariant subset of Z fibering equivariantly over D , with fibers denoted by E_d , $d \in D$. Suppose that for any $d \in D$ there exist an element $\gamma \in \Lambda$, an open set $U \subset Z$ containing E_d , and a constant $c > 1$ such that*

$$(4.6) \quad \text{dist}_Z(\gamma \cdot z, \gamma \cdot E_{d'}) \geq c \text{dist}_Z(z, E_{d'})$$

for all $z \in U$ and $d' \in D$ with $E_{d'} \subset U$. Then the action of Λ on $Z \setminus E$ is cocompact.

Proof of Lemma 4.12. Suppose by contradiction that the action is not cocompact, and let $(\epsilon_n)_{n \in \mathbf{N}}$ be a sequence converging to 0. For any $n \in \mathbf{N}^*$, the set $C_n = \{z \in Z \mid \text{dist}_Z(z, E) \geq \epsilon_n\}$ is compact, hence there exists a Λ -orbit contained in $Z \setminus (C_n \cup E)$; by approaching the supremum of $\text{dist}_Z(\cdot, E)$ on this orbit, we find an element $z_n \in Z$ such that $0 < \text{dist}_Z(z_n, E) \leq \epsilon_n$ and

$$\text{dist}_Z(\gamma \cdot z_n, E) \leq (1 + \epsilon_n) \text{dist}_Z(z_n, E) \quad \forall \gamma \in \Lambda.$$

Up to extracting, we may assume that $(z_n)_{n \in \mathbf{N}^*}$ converges to some $z_\infty \in E$, belonging to a fiber E_d , $d \in D$. Let (γ, U, c) be such that (4.6) holds for all $z \in U$ and $d' \in D$ with $E_{d'} \subset U$. For any $n \in \mathbf{N}^*$, consider $d_n \in D$ such that $\text{dist}_Z(\gamma \cdot z_n, \gamma \cdot E_{d_n})$ is minimal, equal to $\text{dist}_Z(\gamma \cdot z_n, E)$; the sequence $(d_n)_{n \in \mathbf{N}^*}$ converges to d since $(z_n)_{n \in \mathbf{N}^*}$ converges to z_∞ . For large enough n

we have $E_{d_n} \subset U$, and so

$$\begin{aligned} \text{dist}_Z(\gamma \cdot z_n, E) &= \text{dist}_Z(\gamma \cdot z_n, \gamma \cdot E_{d_n}) \\ &\geq c \text{dist}_Z(z_n, E_{d_n}) \geq c \text{dist}_Z(z_n, E) \\ &\geq \frac{c}{1 + \epsilon_n} \text{dist}_Z(\gamma \cdot z_n, E). \end{aligned}$$

This is impossible since $c/(1 + \epsilon_n) > 1$ for large enough n . \square

Proof of Lemma 4.9. If $i = 1$, then Lemma 4.9 is an immediate consequence of Proposition 4.10 and Lemma 4.12 with $(Z, E, D) = (\text{Gr}_q(V), \mathcal{B}_\rho, \partial_\infty \Gamma)$. Suppose now that i is arbitrary and let $\tau_i : \text{GL}_{\mathbf{R}}(V) \rightarrow \text{GL}_{\mathbf{R}}(\bigwedge^i V)$ be the homomorphism coming from the action of $\text{GL}_{\mathbf{R}}(V)$ on the i -th exterior power of V . By Proposition 3.11, the representation ρ is P_i -Anosov if and only if $\tau_i \circ \rho : \Gamma \rightarrow \text{GL}_{\mathbf{R}}(\bigwedge^i V)$ is P_1 -Anosov. The map

$$\begin{aligned} \text{Gr}_q(V) &\longrightarrow \text{Gr}_{(q)}^i\left(\bigwedge^i V\right) \\ W &\longmapsto \bigwedge^i W \end{aligned}$$

is τ_i -equivariant and injective. Moreover, for $E \in \text{Gr}_i(V)$ and $W \in \text{Gr}_q(V)$ we have $E \subset W$ if and only if $\bigwedge^i E \subset \bigwedge^i W$. Therefore Lemma 4.9 for ρ follows from Lemma 4.9 for $\tau_i \circ \rho$ with $i = 1$. \square

Proof of Theorem 4.1. By Proposition 4.3, the action of Γ on $\Omega = \bar{X}_b \setminus \mathcal{N}_\rho$ is properly discontinuous since $\mathcal{N}_\rho = \mathcal{W}_\rho$ for a $P_1(b)$ -Anosov representation. As $\mathcal{N}_\rho = \mathcal{V}_\rho$, cocompactness of the action follows from Lemma 4.8. \square

5. COMPACTIFYING RIEMANNIAN LOCALLY SYMMETRIC SPACES: THE GENERAL CASE

We now use the compactification of Riemannian locally symmetric spaces of indefinite orthogonal groups constructed in Theorem 4.1, Proposition 4.3, and Lemma 4.8 to prove Theorem 1.1 in Section 5.2, Theorem 1.4 in Section 5.4, and a more precise version of Theorem 1.2 in Section 5.5. The case of complex orthogonal groups is investigated in Section 5.6.

5.1. The subalgebra compactification. Let G be a real semisimple Lie group. It acts on its Lie algebra \mathfrak{g} via the adjoint action, preserving the Killing form κ (see Section 3.4). As in Section 4, the Riemannian symmetric space X_κ of the orthogonal group $\text{O}(\kappa)$ admits a realization as an open subset in the Grassmannian $\text{Gr}_{\dim \mathfrak{k}}(V)$, namely as the set of $W \in \text{Gr}_{\dim \mathfrak{k}}(V)$ such that the restriction of κ to $W \times W$ is negative definite. Its closure

$$\bar{X}_\kappa = \{W \in \text{Gr}_{\dim \mathfrak{k}}(\mathfrak{g}) \mid \kappa(x, x) \leq 0 \quad \forall x \in W\}$$

is a compactification of X_κ .

The element $\mathfrak{r}_\emptyset := \mathfrak{k}$ belongs to $X_\kappa \subset \text{Gr}_{\dim \mathfrak{k}}(\mathfrak{g})$ and its stabilizer in G is K . Thus the orbit $\text{Ad}(G) \cdot \mathfrak{r}_\emptyset$ in X_κ identifies with the Riemannian symmetric space $X = G/K$. The closure \bar{X}^{sba} of $X \simeq \text{Ad}(G) \cdot \mathfrak{r}_\emptyset$ in \bar{X}_κ is called the *subalgebra compactification* of X .

Proposition 5.1 ([JL04, Th. 1.1]). *The subalgebra compactification of X is isomorphic to the maximal Satake compactification of X .*

We now describe representatives of the finitely many orbits G -orbits in \bar{X}^{sba} . For $\theta \in \Delta$ the Lie algebra \mathfrak{p}_θ of P_θ has nilpotent radical \mathfrak{u}_θ (see Section 2.3) and a Levi component is

$$\mathfrak{l}_\theta = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma \cap \text{span}(\Delta \setminus \theta)} \mathfrak{g}_\alpha,$$

and $\mathfrak{k}_\theta := \mathfrak{k} \cap \mathfrak{l}_\theta$ is a maximal compact subalgebra of \mathfrak{l}_θ . Set $\mathfrak{r}_\theta = \mathfrak{k}_\theta \oplus \mathfrak{u}_\theta$. (For $\theta = \emptyset$, one has indeed $\mathfrak{r}_\theta = \mathfrak{k}$.)

Lemma 5.2 ([JL04]). *The subalgebra compactification \bar{X}^{sba} is the disjoint union $\bigcup_{\theta \in \Delta} \text{Ad}(G) \cdot \mathfrak{r}_\theta$.*

The kernel of the restriction of κ to \mathfrak{r}_θ is precisely \mathfrak{u}_θ :

Lemma 5.3. *For every $\theta \in \Delta$, one has $\text{Ker}(\kappa|_{\mathfrak{r}_\theta \times \mathfrak{r}_\theta}) = \mathfrak{u}_\theta$. Consequently a nilpotent element Y belongs to \mathfrak{r}_θ if and only if it belongs to \mathfrak{u}_θ .*

Proof. Since $\mathfrak{r}_\theta \subset \mathfrak{p}_\theta$, one has $\mathfrak{u}_\theta = \text{Ker}(\kappa|_{\mathfrak{p}_\theta \times \mathfrak{p}_\theta}) \subset \text{Ker}(\kappa|_{\mathfrak{r}_\theta \times \mathfrak{r}_\theta})$. Furthermore, since the Killing form κ is negative definite in restriction to \mathfrak{k} , the intersection $\text{Ker}(\kappa|_{\mathfrak{r}_\theta \times \mathfrak{r}_\theta}) \cap \mathfrak{k}_\theta$ is trivial, hence $\text{Ker}(\kappa|_{\mathfrak{r}_\theta \times \mathfrak{r}_\theta}) = \mathfrak{u}_\theta$.

A nilpotent element Y satisfies $\kappa(Y, Y) = 0$, hence $Y \in \text{Ker}(\kappa|_{\mathfrak{r}_\theta \times \mathfrak{r}_\theta})$ by Lemma 4.2. \square

5.2. Proof of Theorem 1.1. Let $\rho : \Gamma \rightarrow G$ be a P_θ -Anosov representation. By Proposition 3.12, there exists a homomorphism $\tau : G \rightarrow \text{O}(b)$ such that $\tau \circ \rho : \Gamma \rightarrow \text{O}(b)$ is $P_1(b)$ -Anosov. Let $\Omega = \bar{X}_b \setminus \mathcal{N}_\rho$ be the set given by Theorem 4.1, on which Γ acts properly discontinuously and cocompactly via $\tau \circ \rho$. Let x_0 be a point of X_b whose stabilizer in G is K , and let Y be the $\tau(G)$ -orbit of x_0 : it identifies with $\tau(G)/\tau(K)$. The closure \bar{Y} of Y in \bar{X}_b is a generalized Satake compactification, see Lemma A.7. The group Γ acts properly discontinuously and cocompactly via $\tau \circ \rho$ on $\Omega \cap \bar{Y}$. The quotient $M = (\tau \circ \rho)(\Gamma) \backslash (\Omega \cap \bar{Y})$ thus gives a compactification of $(\tau \circ \rho)(\Gamma) \backslash \tau(G)/\tau(K)$.

By compactness of \bar{X}^{sba} , the action of Γ on $\Omega \times \bar{X}^{sba}$ via $(\tau \times \text{Ad}) \circ \rho$ is properly discontinuous and cocompact. The $\text{Ad}(G)$ -orbit of \mathfrak{r}_θ in \bar{X}^{sba} identifies with $X = G/K$ and is dense. Let Z be the $(\tau \times \text{Ad})(G)$ -orbit of $(x_0, \mathfrak{r}_\theta)$. By Lemma A.8, the closure \bar{Z} of Z in $\bar{X}_b \times \bar{X}^{sba}$ is a generalized Satake compactification of Z . By construction, it dominates the maximal Satake compactification \bar{X}^{sba} of X . Lemma A.9 says that it is a manifold with corners. The group Γ acts properly discontinuously and cocompactly on $(\Omega \times \bar{X}^{sba}) \cap \bar{Z}$ via $(\tau \times \text{Ad}) \circ \rho$. The quotient

$$((\tau \times \text{Ad}) \circ \rho)(\Gamma) \backslash ((\Omega \times \bar{X}^{sba}) \cap \bar{Z})$$

gives a compactification of $((\tau \times \text{Ad}) \circ \rho)(\Gamma) \backslash Z \simeq \rho(\Gamma) \backslash X$ with the required properties. This completes the proof of Theorem 1.1.

5.3. Domains of discontinuity in \bar{X}^{sba} for discrete subgroups.

Theorem 5.4. *Let $X = G/K$ be a Riemannian symmetric space where G is a noncompact real semisimple Lie group. For any discrete subgroup Λ of G there is a closed Λ -invariant subset \mathcal{N} of $\bar{X}^{sba} \setminus X$ such that Λ acts properly discontinuously on $\bar{X}^{sba} \setminus \mathcal{N}$. In particular $\Lambda \backslash (\bar{X}^{sba} \setminus \mathcal{N})$ is a manifold with corners containing $\Lambda \backslash X$ as a dense subset.*

Proof of Theorem 5.4. Let $\text{Ad} : G \rightarrow \text{O}(\kappa)$ be the adjoint action; it has finite kernel. Let Λ be a discrete subgroup of G and $\Omega = \bar{X}_\kappa \setminus \mathcal{W}_{\text{Ad}} \subset \bar{X}_\kappa$ the set given by Proposition 4.4 for $\Gamma = \Lambda$ and $\rho = \text{Ad}$, on which Λ acts properly discontinuously via Ad . The $\text{Ad}(G)$ -orbit of \mathfrak{r}_\emptyset identifies with $X = G/K$. The closure \bar{X}^{sba} of X in \bar{X}_κ is the subalgebra compactification. The group Λ acts properly discontinuously via Ad on $\Omega \cap \bar{X}^{sba}$. The quotient $M = \text{Ad}(\Lambda) \backslash (\Omega \cap \bar{X}^{sba})$ thus is locally modeled on \bar{X}^{sba} and contains $\text{Ad}(\Lambda) \backslash X$. \square

5.4. Topological tameness. Theorem 1.4 is now a direct consequence of the construction of our compactification in Theorem 1.1 and of the following proposition, which is proved in [GGKWb].

Proposition 5.5 ([GGKWb, Prop. 6.1]). *Let X be a real semi-algebraic set and Γ a torsion-free discrete group acting on X by real algebraic homeomorphisms. Suppose Γ acts properly discontinuously and cocompactly on some open subset Ω of X . Let \mathcal{U} be a Γ -invariant real semi-algebraic subset of X contained in Ω (e.g. an orbit of a real algebraic group containing Γ and acting algebraically on X). Then the closure $\bar{\mathcal{U}}$ of \mathcal{U} in X is real semi-algebraic and $\Gamma \backslash (\bar{\mathcal{U}} \cap \Omega)$ is compact and has a triangulation such that $\Gamma \backslash (\partial \mathcal{U} \cap \Omega)$ is a finite union of simplices. If \mathcal{U} is a manifold, then $\Gamma \backslash \mathcal{U}$ is topologically tame.*

5.5. Compactifications modeled on the maximal Satake compactification. We now prove the following theorem, which, together with Proposition 5.1, implies Theorem 1.2 in the case that G is simple.

Theorem 5.6. *Let G be a real simple Lie group. Let $\alpha_G \in \Delta$ be the simple restricted root given by Proposition 3.14 (see Table 1) and $d = \dim_{\mathbf{R}} \mathfrak{g}_{\chi_G}$, where $\chi_G \in \Sigma^+$ is the highest restricted root. Let Γ be a word hyperbolic group and $\rho : \Gamma \rightarrow G$ a $P_{\{\alpha_G\}}$ -Anosov representation. Let $\xi_d : \partial_\infty \Gamma \rightarrow \mathcal{F}_d(\kappa)$ be the boundary map of $\text{Ad} \circ \rho$ (see Proposition 3.14) and*

$$\mathcal{N}_\rho = \bigcup_{\eta \in \partial_\infty \Gamma} \{W \in \bar{X}^{sba} \mid \xi_d(\eta) \subset W\}.$$

Then $\Omega := \bar{X}^{sba} \setminus \mathcal{N}_\rho$ contains $X = G/K$ and the action of Γ on Ω is properly discontinuous and cocompact.

Proof. By Proposition 4.3, the action of Γ via $\text{Ad} \circ \rho$ on $\bar{X}_\kappa \setminus \mathcal{W}_{\text{Ad} \circ \rho}$ is properly discontinuous. Thus the action of Γ on $\bar{X}^{sba} \setminus (\mathcal{W}_{\text{Ad} \circ \rho} \cap \bar{X}^{sba})$ is properly discontinuous.

Let us prove that $\mathcal{N}_\rho = \mathcal{W}_{\text{Ad} \circ \rho} \cap \bar{X}^{sba}$ and that \mathcal{N}_ρ does not intersect X . The inclusion $\mathcal{N}_\rho \subset \mathcal{W}_{\text{Ad} \circ \rho} \cap \bar{X}^{sba}$ is obvious. Let now $W \in \mathcal{W}_{\text{Ad} \circ \rho} \cap \bar{X}^{sba}$, i.e. there is $\eta \in \partial_\infty \Gamma$ such that $W \cap \xi_d(\eta) \neq \emptyset$. By Proposition 3.14 $\xi_d(\eta) \in \text{Ad}(G) \cdot \mathfrak{g}_{\chi_G}$. Lemma 5.7 implies that $W \not\subset X$, hence $\mathcal{N}_\rho \cap X = \emptyset$, and that $\xi_d(\eta) \subset W$, i.e. $W \in \mathcal{N}_\rho$. Since $\mathcal{V}_{\text{Ad} \circ \rho} \cap \bar{X}^{sba} = \mathcal{N}_\rho$, Lemma 4.8 implies that the action of Γ on Ω is cocompact. \square

Lemma 5.7. *Let $W \in \bar{X}^{sba}$ and $L \in \text{Ad}(G) \cdot \mathfrak{g}_{\chi_G} \subset \text{Gr}_d(\mathfrak{g})$. If $L \cap W \neq \{0\}$, then $W \not\subset X$ and $L \subset \text{Ker}(\kappa|_{W \times W}) \subset W$.*

Proof of Lemma 5.7. Since the elements of L are nilpotent, the hypothesis implies that $\text{Ker}(\kappa|_{W \times W}) \neq \{0\}$, hence $W \not\subset X$.

By Lemma 5.2 there exist $\theta \subset \Delta$ and $h \in G$ such that $W = \text{Ad}(h) \cdot \mathfrak{r}_\theta$ and there exists $h' \in G$ such that $L = \text{Ad}(h') \cdot \mathfrak{g}_{\chi_G}$. Lemma 5.3 implies that the intersection of $\text{Ker}(\kappa|_{W \times W}) = \text{Ad}(h) \cdot \mathfrak{u}_\theta$ and L is nontrivial. We need to prove that $L \subset \text{Ad}(h) \cdot \mathfrak{u}_\theta$.

The element $g = h^{-1}h'$ admits a Bruhat decomposition $g = p\tilde{w}p'$, i.e. p and p' are in the minimal parabolic subgroup P_Δ and \tilde{w} belongs to $N_K(\mathfrak{a})$ (see e.g. [Kna02, Th. 7.40]). Let w be the class of \tilde{w} in $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. Thus

$$L = \text{Ad}(h)\text{Ad}(g) \cdot \mathfrak{g}_{\chi_G} = \text{Ad}(h)\text{Ad}(p)\text{Ad}(\tilde{w}) \cdot \mathfrak{g}_{\chi_G}$$

(since $\text{Ad}(p') \cdot \mathfrak{g}_{\chi_G} = \mathfrak{g}_{\chi_G}$) and $L = \text{Ad}(hp) \cdot \mathfrak{g}_{w \cdot \chi_G}$. Also $\text{Ad}(h) \cdot \mathfrak{u}_\theta = \text{Ad}(hp) \cdot \mathfrak{u}_\theta$. Hence the Lie algebra \mathfrak{u}_θ has a nontrivial intersection with $\mathfrak{g}_{w \cdot \chi_G}$. Since the root space decomposition is direct, this is possible if and only if $\mathfrak{g}_{w \cdot \chi_G} \subset \mathfrak{u}_\theta$. This implies that $L \subset \text{Ad}(h) \cdot \mathfrak{u}_\theta \subset W$. \square

Theorem 5.8. *Let G be a real semisimple Lie group and let $\Phi \subset \Delta$ be the set consisting of the simple restricted roots $\alpha_{G'}$ given by Proposition 3.14 (see Table 1) for all the simple factors G' of G . Let $\theta \subset \Delta$ be nonempty with $\Phi \cap \theta \neq \emptyset$. For any word hyperbolic group Γ and any P_θ -Anosov representation $\rho : \Gamma \rightarrow G$, the Riemannian locally symmetric space $\rho(\Gamma) \backslash G/K$ admits a compactification modeled on the maximal Satake compactification of G/K .*

Proof of Theorem 5.8. There is a simple factor G' of G such that the projection ρ' of ρ to G' is $P_{\{\alpha_{G'}\}}$ -Anosov. The maximal Satake compactification \bar{X} of $X = G/K$ is the product $\bar{X}' \times \bar{X}''$ of the maximal Satake compactification of G'/K' and of the maximal Satake compactification of the Riemannian symmetric space associated with the other factors of G (see Section A.1). By Theorem 5.6 there is an open set $\Omega' \subset \bar{X}'$ containing the Riemannian symmetric space of G' such that the action of Γ via ρ on Ω' is properly discontinuous and cocompact. It follows that the action of Γ via ρ on $\Omega' \times \bar{X}''$ is properly discontinuous and cocompact. \square

5.6. Complex orthogonal groups. Let $O(b^{\mathbf{C}})$ be the orthogonal group of a nondegenerate complex symmetric bilinear form $b^{\mathbf{C}}$ on a complex vector space V of dimension n .

The parabolic subgroup $P_1(b^{\mathbf{C}})$ is defined as the stabilizer of the line $\mathbf{C}e_1$ where $b^{\mathbf{C}}(e_1, e_1) = 0$; it is conjugate to its opposite. The homogeneous space $\mathcal{F}_1(b^{\mathbf{C}}) = O(b^{\mathbf{C}})/P_1(b^{\mathbf{C}})$ is the set of $b^{\mathbf{C}}$ -isotropic complex lines in V .

The Riemannian symmetric space $X_{b^{\mathbf{C}}}$ of $O(b^{\mathbf{C}})$ can be realized as a subset in the Grassmanian $\text{Gr}_n^{\mathbf{R}}(V)$ of n -dimensional real subspaces of V , with compactification

$$(5.1) \quad \bar{X}_{b^{\mathbf{C}}} = \{W \in \text{Gr}_n^{\mathbf{R}}(V) \mid b^{\mathbf{C}}(W \times W) \subset \mathbf{R}, \text{ and } b^{\mathbf{C}}(w, w) \leq 0 \forall w \in W\}.$$

In fact $O(b^{\mathbf{C}}) \subset O(b)$ where $b = \text{Re}(b^{\mathbf{C}})$ and there is an inclusion $X_{b^{\mathbf{C}}} \subset X_b$. The compactification $\bar{X}_{b^{\mathbf{C}}}$ is precisely the closure of $X_{b^{\mathbf{C}}}$ in \bar{X}_b . It is isomorphic to a minimal Satake compactification if n is odd and to a generalized Satake compactification if n is even.

Theorem 5.9. *Let $b^{\mathbf{C}}$ be a nondegenerate complex symmetric bilinear form on a complex vector space V of dimension $n \geq 3$, let Γ be a word hyperbolic group, and let $\rho : \Gamma \rightarrow O(b^{\mathbf{C}})$ be a $P_1(b^{\mathbf{C}})$ -Anosov representation with*

boundary map $\xi : \partial_\infty \Gamma \rightarrow \mathcal{F}_1(b^{\mathbf{C}})$. Let

$$\mathcal{N}_\rho = \bigcup_{\eta \in \partial_\infty \Gamma} \{W \in \bar{X}_{b^{\mathbf{C}}} \mid \xi(\eta) \subset W\}.$$

Then the action of Γ on $\Omega = \bar{X}_{b^{\mathbf{C}}} \setminus \mathcal{N}_\rho$ is properly discontinuous and cocompact. The set Ω contains the Riemannian symmetric space $X_{b^{\mathbf{C}}}$ and $\rho(\Gamma) \backslash \Omega$ is a smooth compactification of $\rho(\Gamma) \backslash X_{b^{\mathbf{C}}}$.

Proof of Theorem 5.9. Denote by τ the natural injection of $O(b^{\mathbf{C}})$ into $O(b)$ where $b = \operatorname{Re}(b^{\mathbf{C}})$. The representation $\tau \circ \rho$ is $P_2(b)$ -Anosov (see Proposition 3.12 and Lemma 3.13).

By Proposition 4.3 the action of Γ on $\bar{X}_b \setminus \mathcal{W}_{\tau \circ \rho}$ is properly discontinuous and thus the action of Γ via ρ on $\bar{X}_{b^{\mathbf{C}}} \setminus (\mathcal{W}_{\tau \circ \rho} \cap \bar{X}_{b^{\mathbf{C}}})$ is as well properly discontinuous. Lemma 5.10 implies that $\mathcal{W}_{\tau \circ \rho} \cap \bar{X}_{b^{\mathbf{C}}} = \mathcal{N}_\rho$.

By Lemma 4.8 the action of Γ on $\bar{X}_b \setminus \mathcal{V}_{\tau \circ \rho}$ is cocompact and thus the action of Γ via ρ on $\bar{X}_{b^{\mathbf{C}}} \setminus (\mathcal{V}_{\tau \circ \rho} \cap \bar{X}_{b^{\mathbf{C}}})$ is as well cocompact. Since $\mathcal{V}_{\tau \circ \rho} \cap \bar{X}_{b^{\mathbf{C}}} = \mathcal{N}_\rho$ the theorem follows. \square

Lemma 5.10. *If $W \in \bar{X}_{b^{\mathbf{C}}}$ then $\operatorname{Ker}(b^{\mathbf{C}}|_{W \times W})$ is a \mathbf{C} -vector subspace of \mathbf{C}^n . If $L \in \mathcal{F}_1(b^{\mathbf{C}})$, then $L \cap W \neq \{0\} \Leftrightarrow L \subset W$.*

Proof of Lemma 5.10. The kernel $\operatorname{Ker}(b^{\mathbf{C}}|_{W \times W})$ is a real vector space, we need to prove that it is stable by multiplication by $\sqrt{-1}$. Let $z \in \operatorname{Ker}(b^{\mathbf{C}}|_{W \times W})$, then $b^{\mathbf{C}}(z, z) = 0$ and $b^{\mathbf{C}}(x, z) = 0$ for all $x \in W$. Set $y = \sqrt{-1}z$ and $b = \operatorname{Re}(b^{\mathbf{C}})$. One has $b(y, y) = 0$ and $b(x, y) = 0$ for all $x \in W$. By Lemma 4.2, $y \in W$ and y belongs to $\operatorname{Ker}(b^{\mathbf{C}}|_{W \times W})$.

If $L \cap W \neq \{0\}$ then $L \cap \operatorname{Ker}(b^{\mathbf{C}}|_{W \times W}) \neq \{0\}$ and $L \subset \operatorname{Ker}(b^{\mathbf{C}}|_{W \times W}) \subset W$ since L has complex dimension 1. \square

APPENDIX A. SATAKE COMPACTIFICATIONS

A.1. Satake compactifications. In this section we briefly review the construction of the Satake compactification of a Riemannian symmetric space $X = G/K$, which was originally defined in [Sat60]. We denote by \mathcal{H}_n the space of Hermitian $(n \times n)$ matrices over \mathbf{C} .

Let $\tau : G \rightarrow \operatorname{PSL}(n, \mathbf{C})$ be an irreducible projective representation with finite kernel. We may assume that $\tau(K) \subset \operatorname{PSU}(n)$. By definition, the *Satake compactification* \bar{X}_τ of X associated with τ is the closure in $\mathbf{P}(\mathcal{H}_n)$ of the image of X under the embedding $X \rightarrow \mathbf{P}(\mathcal{H}_n)$ given by $gK \mapsto \mathbf{R}(\tau(g)\tau(g)^*)$, where M^* is the transpose-conjugate of a matrix M .

The structure of the Satake compactification \bar{X}_τ as a G -space only depends on the support

$$\theta_\tau = \{\alpha \in \Delta \mid (\chi_\tau, \alpha) > 0\}$$

of the highest weight χ_τ of the irreducible representation τ . Satake compactifications have the following properties:

- (1) The compactification \bar{X}_τ has finitely many G -orbits, including a unique open G -orbit, namely $X = G/K$, and a unique closed orbit, which identifies with G/P_{θ_τ} .
- (2) If $\theta_{\tau'} \subset \theta_\tau$ then there exists a continuous (hence proper) surjective G -equivariant map $\pi_{\tau, \tau'} : \bar{X}_\tau \rightarrow \bar{X}_{\tau'}$.

- (3) Every Satake compactification of a product is a product of Satake compactifications.

By (2), the Satake compactification \bar{X}_τ for $\theta_\tau = \Delta$ surjects onto any Satake compactification $\bar{X}_{\tau'}$ of X ; it is called the *maximal* Satake compactification of X . On the other hand, Satake compactifications of the form \bar{X}_τ for $\sharp\theta_\tau = 1$ are called *minimal* Satake compactifications. The maximal Satake compactification of X is a manifold with corners [BJ06, Prop. I.19.27]. The set θ_τ will be called the *support* of the Satake compactification \bar{X}_τ .

A.2. Orbits description. The finitely many orbits of a Satake compactification are described by the some combinatorial data associated with the irreducible representation τ . As our convention for parabolic groups is opposite to [BJ06], the terminology and description of the orbits and the stabilizers has to be adapted from the classical case.

For any subset $\theta \subset \Delta$ the parabolic algebra \mathfrak{p}_θ is the direct sum

$$\mathfrak{p}_\theta = \mathfrak{u}_\theta \oplus \mathfrak{a}_\theta \oplus \mathfrak{m}_\theta$$

where $\mathfrak{a}_\theta = \bigcap_{\alpha \in \Delta \setminus \theta} \text{Ker}(\alpha)$, $\mathfrak{m}_\theta = \mathfrak{z}_\mathfrak{k}(\mathfrak{a}) \oplus \mathfrak{a}^\theta \oplus \bigoplus_{\alpha \in \Sigma \cap \text{span}(\Delta \setminus \theta)} \mathfrak{g}_\alpha$, and $\mathfrak{a}^\theta = \mathfrak{a} \cap (\mathfrak{a}_\theta)^\perp$. The corresponding Lie groups are denoted by U_θ , A_θ and M_θ . The map $U_\theta \times A_\theta \times M_\theta \rightarrow P_\theta \mid (u, a, m) \mapsto uam$ is a diffeomorphism and we will simply write in the sequel $P_\theta = U_\theta A_\theta M_\theta$.

A subset $\theta \subset \Delta$ will be said τ -admissible if the graph with vertex sets $(\Delta \setminus \theta) \cup \{\chi_\tau\}$ and edges between every pairs with a non zero scalar products is connected. (One usually says that $\Delta \setminus \theta$ is τ -connected.)

For such a subset let $\theta^\vee = \{\alpha \in \Delta \mid \exists \beta \in (\Delta \setminus \theta) \cup \{\chi_\tau\}, (\alpha, \beta) \neq 0\}$ be the set of the elements of Δ being non-orthogonal to an element in $(\Delta \setminus \theta) \cup \{\chi_\tau\}$. The set $\theta^\ddagger = \theta \cap \theta^\vee$ is called the τ -nucleus of θ . (The usual terminology says that $\Delta \setminus \theta^\cap$ is the τ -saturation of Δ .) Note that M_{θ^\ddagger} is the almost product of M_θ and M_{θ^\vee} and that these two last groups commute.

The Satake compactification \bar{X}_τ admits the following description:

(a) it is the disjoint union of the G -orbits of points x_θ over the τ -admissible sets $\theta \subset \Delta$;

(b) the stabilizer of x_θ is the product $U_{\theta^\ddagger} A_{\theta^\ddagger} M_{\theta^\vee} (K \cap M_\theta)$; in particular is contained in P_{θ^\ddagger} ;

(c) the orbit $G \cdot x_\theta$ fibers over the flag manifold $G/P_{\theta^\ddagger} = \mathcal{F}_{\theta^\ddagger}$ and the fibers are isomorphic to $M_\theta/(K \cap M_\theta)$, i.e. to the Riemannian symmetric space associated with the reductive group M_θ ;

(d) the orbit $G \cdot x_\emptyset$ is the copy of the Riemannian symmetric space G/K ;

(e) the unique closed orbit is $G \cdot x_\Delta$.

In order to describe the topology on \bar{X}_τ , it is enough to understand the closure of the Weyl chamber. Let $(H_n)_{n \in \mathbf{N}}$ be a sequence in $\bar{\mathfrak{a}}^+$, then the sequence $(\exp(H_n) \cdot x_\emptyset)_{n \in \mathbf{N}}$ converges in \bar{X}_τ if and only if there exists a τ -admissible set $\theta \subset \Delta$ such that

- (i) For each $\alpha \in \Delta \setminus \theta$ the sequence $(\langle \alpha, H_j \rangle)_{n \in \mathbf{N}}$ converges to some $t_\alpha \in \mathbf{R}$ and
- (ii) for every τ -admissible set $\theta' \subsetneq \theta$ there exists $\alpha \in \theta \setminus \theta'$ such that $\lim_{n \rightarrow \infty} \langle \alpha, H_n \rangle = +\infty$.

Furthermore, if H is the unique element of \mathfrak{a}^θ such that $\langle \alpha, H \rangle = t_\alpha$ for all $\alpha \in \Delta \setminus \theta$, then $\lim_{n \rightarrow \infty} \exp(H_n) \cdot x_\theta = \exp(H) \cdot x_\theta$.

A.3. A minimal Satake compactification of X_b . Recall that for real symmetric bilinear forms b , a compactification \bar{X}_b of the Riemannian symmetric space X_b of $O(b)$ was constructed in Section 4. Similarly, for complex symmetric bilinear forms $b^{\mathbf{C}}$, a compactification $\bar{X}_{b^{\mathbf{C}}}$ of the Riemannian symmetric space $X_{b^{\mathbf{C}}}$ of $O(b^{\mathbf{C}})$ was constructed in Section 5.6.

Proposition A.1. *Let b be a nondegenerate symmetric bilinear form of signature (p, q) on a real vector space V . If $p > q$, then \bar{X}_b is a minimal Satake compactification of X_b .*

When $p = q$, \bar{X}_b is not a Satake compactification, see Example A.4.

Proof. The natural representation $\tau : O(b) \rightarrow \mathrm{GL}_{\mathbf{R}}(\wedge^q V)$ identifies X_b with a subset of $\mathbf{P}(\wedge^q V)$, and \bar{X}_b with the closure of this set in $\mathbf{P}(\wedge^q V)$. Since $\wedge^q V$ is an irreducible representation of $O(b)$, the result of [Kor09] applies to show that \bar{X}_b is a Satake compactification of X_b and that its support is $\{\alpha \in \Delta \mid \langle \chi_\tau, \alpha \rangle \neq 0\}$ if χ_τ is the highest weight of τ . This support is thus $\{\alpha_q\}$ and the compactification is a minimal Satake compactification. \square

Proposition A.2. *Let $b^{\mathbf{C}}$ be a nondegenerate symmetric complex bilinear form on a complex vector space V of dimension n . If n is odd, then $\bar{X}_{b^{\mathbf{C}}}$ is a minimal Satake compactification of $X_{b^{\mathbf{C}}}$.*

Proof. Write $n = 2m + 1$. There is a basis (e_1, \dots, e_n) of V such that $b^{\mathbf{C}}(x, y) = \sum_{i=1}^n x_i y_{n-i+1}$ for every $x = \sum_{i=1}^n x_i e_i$ and $y = \sum_{i=1}^n y_i e_i$ of V . One can take $K = U(n)$ and \mathfrak{a} the diagonal matrices in that basis. Namely

$$\mathfrak{a} = \{\mathrm{diag}(\lambda_1, \dots, \lambda_m, 0, -\lambda_m, \dots, -\lambda_1) \mid \lambda_1, \dots, \lambda_m \in \mathbf{R}\}.$$

For $i = 1, \dots, m$, set $\langle \varepsilon_i, \mathrm{diag}(\lambda_1, \dots, \lambda_m, 0, -\lambda_m, \dots, -\lambda_1) \rangle = \lambda_i$. The restricted root system is

$$\Sigma = \{\pm \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq m\} \cup \{\pm \varepsilon_i \mid 1 \leq i \leq m\}.$$

A system of simple restricted roots is $\Delta = \{\alpha_1, \dots, \alpha_m\}$ where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq m-1$ and $\alpha_m = \varepsilon_m$. We will show that $\bar{X}_{b^{\mathbf{C}}}$ satisfies the above axiomatic description of Satake compactification whose support is $\{\alpha_m\}$. The admissible sets are $\theta_0 = \emptyset$, $\theta_1 = \{\alpha_1\}$, \dots , $\theta_m = \Delta$. One checks that $\theta_0^\vee = \Delta$, $\theta_0^\dagger = \emptyset$ and $\theta_i^\vee = \{\alpha_i, \dots, \alpha_m\}$, $\theta_i^\dagger = \{\alpha_i\}$ for $1 \leq i \leq m$.

We first consider the orbits description of $\bar{X}_{b^{\mathbf{C}}}$. By Lemma 5.10, for any $W \in \bar{X}_{n, \mathbf{C}}$, the set $\mathrm{Ker}(b^{\mathbf{C}}|_{W \times W})$ is a \mathbf{C} -vector subspace of V . Applying Witt's theorem, one shows that the sets

$$\mathcal{U}_i = \{W \in \bar{X}_{b^{\mathbf{C}}} \mid \dim_{\mathbf{C}} \mathrm{Ker}(b^{\mathbf{C}}|_{W \times W}) = i\},$$

$i = 0, \dots, m$, are the $O(b^{\mathbf{C}})$ -orbits in $\bar{X}_{b^{\mathbf{C}}}$. For $0 \leq i \leq m$, the real vector space

$$\begin{aligned} W_{\theta_i} = \mathrm{Span}_{\mathbf{R}}(e_1, \sqrt{-1}e_1, \dots, e_i, \sqrt{-1}e_i, e_{i+1} - e_{n-i}, \sqrt{-1}(e_{i+1} + e_{n-i}), \\ \dots, e_m - e_{m+2}, \sqrt{-1}(e_m + e_{m+2}), \sqrt{-1}e_{m+1}) \end{aligned}$$

belongs to \mathcal{U}_i and its stabilizer in $O(b^{\mathbf{C}})$ is precisely the group $U_{\theta_i^\dagger} A_{\theta_i^\dagger} M_{\theta_i^\vee}(K \cap M_{\theta_i})$ described above and contained in the parabolic subgroup $P_{\theta_i^\dagger}$.

For an element H in \mathfrak{a} one has

$$\begin{aligned} \exp(H) \cdot W_{\theta_0} &= \text{Span}_{\mathbf{R}}(e_1 - e^{-2\langle \varepsilon_1, H \rangle} e_n, \sqrt{-1}(e_1 + e^{-2\langle \varepsilon_1, H \rangle} e_n), \\ &\quad \dots, e_m - e^{-2\langle \varepsilon_m, H \rangle} e_{m+2}, \sqrt{-1}(e_m + e^{-2\langle \varepsilon_m, H \rangle} e_{m+2}), \sqrt{-1}e_{m+1}). \end{aligned}$$

From this for a sequence $(H_k)_{k \in \mathbf{N}}$ in $(\mathfrak{a}^+)^{\mathbf{N}}$, the sequence $(\exp(H_k) \cdot W_{\theta_0})_{k \in \mathbf{N}}$ converges if and only if there exists $0 \leq i \leq m$ for which $(\langle \varepsilon_j, H_k \rangle)_{k \in \mathbf{N}}$ goes to infinity for $j \leq i$ and has a limit in \mathbf{R} for $j > i$. This is equivalent to $(\langle \alpha_j, H_k \rangle)_{k \in \mathbf{N}}$ goes to infinity for $j \leq i$ and has a limit in \mathbf{R} for $j > i$. It follows that the closure of the Weyl chamber satisfies the above description. \square

Remark A.3. For even n , the same analysis can be used to show that $\bar{X}_{b^{\mathbf{C}}}$ is not a Satake compactification.

A.4. Generalized Satake compactifications. The classical notion of Satake compactification does not behave well with respect to totally geodesic subspaces: If \bar{X}_τ is a Satake compactification of X and if $Y \subset X$ is a totally geodesic subsymmetric space, then the closure of Y in \bar{X}_θ is not always a Satake compactification of Y .

Example A.4. Consider a real vector space V' equipped with a symmetric bilinear form b' of signature $(p, p+1)$ and let $V \subset V'$ be a subspace such that $b = b'|_{V \times V}$ is of signature (p, p) . Then the closure of X_b inside $\bar{X}_{b'}$ is not a Satake compactification. Indeed this closure is \bar{X}_b which contains two closed $O(b)_0$ -orbits, contradicting the above description of Section A.2.

In order to obtain a class of compactifications which have this functoriality properties we will have to consider a small generalization of Satake compactifications, which we call *generalized Satake compactifications*. The only difference is that we allow the representation τ to be a sum of irreducible representations.

Remark A.5. Compare with [Sat60, § 5.3], where Satake considers reducible representations, but then takes the closure in $\mathbf{P}(\mathcal{H}_{n_1}) \times \dots \times \mathbf{P}(\mathcal{H}_{n_k})$ instead of $\mathbf{P}(\mathcal{H}_{\sum_{i=1}^k n_i})$.

Definition A.6. Let G be a semisimple Lie group and $\tau : G \rightarrow \text{SL}(n, \mathbf{C})$ a faithful projective representation with $\tau(K) \subset \text{PSU}(n)$. The *generalized Satake compactification* of $X = G/K$ associated with τ is the closure of the image of X under the map $X \rightarrow \mathbf{P}(\mathcal{H}_n)$ given by $gK \mapsto \mathbf{R}(\tau(g)\tau(g)^*)$.

Lemma A.7. Let $X = L/K$ be a Riemannian symmetric space, \bar{X} a generalized Satake compactification of X , and $Y = H/(K \cap H) \hookrightarrow X$ a totally geodesic subsymmetric space of X . Then the closure of Y in \bar{X} is a generalized Satake compactification of Y .

Proof of Lemma A.7. Let $\tau : L \rightarrow \text{SL}(n, \mathbf{C})$ be a representation with finite kernel such that $\bar{X} = \bar{X}_\tau$ and let $\phi : H \rightarrow L$ be the Lie group homomorphism associated with the embedding $Y \hookrightarrow X$. Then the closure of Y is the generalized Satake compactification associated with $\tau \circ \phi : H \rightarrow \text{SL}(n, \mathbf{C})$. \square

From this we deduce the following.

Lemma A.8. *Let $X = G/K$ be a Riemannian symmetric space, $\tau_1 : G \rightarrow \mathrm{SL}(n_1, \mathbf{C})$ a representation with $\tau_1(K) \subset \mathrm{U}(n_1)$ and $\tau_2 : G \rightarrow \mathrm{SL}(n_2, \mathbf{C})$ a representation with finite kernel and with $\tau_2(K) \subset \mathrm{U}(n_2)$. Let \bar{X}_i , $i = 1, 2$ be the closure of $\tau_i(G)/\tau_i(K)$ in $\mathbf{P}(\mathcal{H}_{n_i})$ so that \bar{X}_2 is a generalized Satake compactification. Let*

$$\begin{aligned} \psi : X &\longrightarrow \bar{X}_1 \times \bar{X}_2 \\ g \cdot K &\longmapsto (\mathbf{R}(\tau_1(g)\tau_1^*(g)), \mathbf{R}(\tau_2(g)\tau_2^*(g))). \end{aligned}$$

Then the closure of $\psi(X)$ is a generalized Satake compactification.

Proof of Lemma A.8. Apply Lemma A.7 with $L = \mathrm{SL}(n_1, \mathbf{C}) \times \mathrm{SL}(n_2, \mathbf{C})$ and $H = (\tau_1, \tau_2)(G)$. \square

We say that a compactification \bar{X}_1 *dominates* a compactification \bar{X}_2 if there is a continuous G -equivariant map $\bar{X}_1 \rightarrow \bar{X}_2$, such a map is necessarily surjective and proper.

Lemma A.9. *Let \bar{X} be a generalized Satake compactification dominating the maximal Satake compactification, then \bar{X} is a manifold with corners.*

Proof of Lemma A.9. Let $\psi : \bar{X} \rightarrow \bar{X}^{max}$ be the continuous G -equivariant map. It is easily seen that the closure \bar{F} of F the $\exp(\mathfrak{a})$ -orbit of the base point x_0 in $G/K \subset \bar{X}$ is a manifold with corners. Let $\bar{F}^+ \subset \bar{F}$ be the closure of $\exp(\bar{\mathfrak{a}}^+) \cdot x_0$. Let x be any point of \bar{X} . Using the Cartan decomposition of G one can assume that $x \in \bar{F}^+$. Furthermore, using the Iwasawa decomposition, one deduces that the map $f : U_\emptyset^- \times \bar{F} \rightarrow \bar{X}$ is surjective in a neighborhood of (e, x) into a neighborhood of x . Since the corresponding map $U_\emptyset^- \times \psi(\bar{F}) \rightarrow \bar{X}^{max}$ is a local diffeomorphism we deduce that f is also injective. As $U_\emptyset^- \times \bar{F}$ is a manifold with corners, the lemma follows. \square

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